# Topics in Algebra Notes

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# Contents

# 0 Introduction

These are my notes for the Topics in Algebra course taught in the Fall of 2019 by Dr Ana Lecuona.

If you find any errors please tell me (or email me at asnadiga@gmail.com).

# 1 First Isomorphism Theorem

**Theorem 1.0.1: First Isomorphism Theorem**
If $\varphi : G \to H$ is a homomorphism of groups then $G/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$

**Definition 1.0.2**
Let $G$ be a group and $H_1$ and $H_2$ be subgroups. Then the *join* of $H_1$ and $H_2$ is

$$H_1 \vee H_2 = \bigcap_{H_1 H_2 \leq G'} G'.$$

The join of two subgroups is the smallest subgroup containing both.

**Lemma 1.0.3**
Let $H$ and $N$ be subgroups of a group $G$, and let $N$ be normal in $G$. Then $H \vee N = HN$. Moreover, if $H$ is also normal in $G$, then $HN$ is normal in $G$.

*Proof.* We will show that $HN$ is a subgroup. Since $H \vee N$ is the smallest subgroup that contains $HN$, this will prove the proposition. So let $x = h_1 n_1 \in HN$ and $y = h_2 n_2 \in HN$. We msut show that $xy^{-1} \in HN$. $xy^{-1} = h_1 n_1 n_2^{-1} h_2^{-1}$. Since $N$ is a normal subgroup, this is $h_1 h'(n_1 n_2^{-1})$ for some $h'$, and this is an element of $HN$.
Now suppose that $H$ is normal in $G$. Then for any $g \in G$, $g(hn)g^{-1} = ghg^{-1}gng^{-1}$, and since $H$ and $N$ are normal they are closed under conjugation and this is an element of $HN$. □

**Theorem 1.0.4: Second Isomorphism Theorem**
Let $H$ and $N$ be subgroups of a group $G$, and let $N$ be normal in $G$. Then $H/(H \cap N) \cong NH/N$.

*Proof.* First we must show that $H/(H \cap N)$ and $NH/N$ are groups at all (done by showing that the subgroups are normal in the appropriate groups). Let $i : H \to HN$ by $h \mapsto he_N$, and let $q : HN \to HN/N$ by $hn \mapsto (hn)N$. Now let $f = q \circ i : H \to HN/N$. Then $\mathrm{Im}(f) = HN/N$ and $\mathrm{Ker}(f) = H \cap N$. Now we apply the first isomorphsim theorem to get the result. □

**Theorem 1.0.5: Third Isomorphism Theorem**
If $H$ and $K$ are normal subgroups of a group $G$, and $K \subseteq H$, then $(G/K)/(H/K) \cong G/H$

**Theorem 1.0.6: Correspondence Theorem**
Let $K$ be a normal subgroup of a group $G$. There is a one to one correspondence between (normal) subgroups $H$ of $G$ that contain $K$ and (normal) subgroups of $G/K$. This correspondence is given by $H \mapsto H/K$.

# 2 Series of Groups

**Definition 2.0.1**
A group $G$ is *simple* if it has no non-trivial normal subgroups.

Note that this definition is equivalent to saying that $G$ has no non-trivial factor groups.

**Definition 2.0.2**
Consider a finite series of subgroups of a group $G$, $\{1_G\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$.

- If $H_i \trianglelefteq H_{i+1}$ for all $i$ the series is a a *subnormal series.*
- If $H_i \trianglelefteq G$ for all $i$ then the series if a *normal series.*
- If a series is normal or subnormal, then the groups $H_{i+1}/H_i$ are called *factor groups* of $G$.
- If all the factor groups of a subnormal series are simple it is a *composition series.*
- If all the factor groups of a normal series are simple then it is a *principal series.*

Note that if a group is Abelian then all series are normal and principal series.

**Example 2.0.3**

- In $(\mathbb{Z}, +)$, the series $\{0\} \leq 8\mathbb{Z} \leq 4\mathbb{Z} \leq \mathbb{Z}$ is a subnormal (or normal since $\mathbb{Z}$ Abelian) but not principal series.
- $D_8$, the dihedral group with 8 elements has the following subnormal series $\{1\} \leq \langle s \rangle \leq \langle r^2, s \rangle \leq D_8$. To check that this series is subnormal note that the index of each subgroup in the following subgroup is 2.
- In $(\mathbb{Z}_{15}, +)$, both of the following series are normal (beacuse the group is Abelian): $\{0\} \leq \angle 5 \rangle \leq \mathbb{Z}_{15}$, and $\{0\} \leq \langle 3 \rangle \leq \mathbb{Z}_{15}$.

**Definition 2.0.4**
$\{K_j\}$ is a refinement of a series of subgroups $\{H_i\}$ if $\{H_i\} \subseteq \{K_j\}$.

**Definition 2.0.5 (T)**
o subnormal series $\{H_i\}$ and $\{K_j\}$ are isomorphic if there exists a bijection between $\{K_{i+1}/K_i\}$ and $\{H_{i+1}/H_i\}$ such that corresponding factor groups are isomorphic.

The third example from above is an example of isomorphic series.

**Theorem 2.0.6: Schreier's Theorem**
The subnormal series of the same group admit isomorphic refinements.

We do not prove this theorem, but the proof of the theorem is constructive, meaning that it gives the refinements of each series that are isomorphic.

**Example 2.0.7**
The series $\{0\} \leq 8\mathbb{Z} \leq 4\mathbb{Z} \leq \mathbb{Z}$, and $\{0\} \leq 9\mathbb{Z} \leq \mathbb{Z}$ can be refined to $\{0\} \leq 72\mathbb{Z} \leq 8\mathbb{Z} \leq 4\mathbb{Z} \leq \mathbb{Z}$ and $\{0\} \leq 72\mathbb{Z} \leq 18\mathbb{Z} \leq 9\mathbb{Z} \leq \mathbb{Z}$, respectively. These refinements are isomorphic.

**Proposition 2.0.8**
$\mathbb{Z}$ has no composition series.

*Proof.* Suppose for contradiction that $\{0\} \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = \mathbb{Z}$. Then we know that all normal subgroups are of the form $m\mathbb{Z}$ for some $m \in \mathbb{Z}$. This is true in particular for $H_1$. But then $H_1/\{0\} \cong \mathbb{Z}$, and this is not simple, contradicting the assumption that we have a composition series. □

**Theorem 2.0.9: Jordan-Holder**
Let $G$ be a group that admits a composition series. Then any two composition series are isomorphic.

*Proof.* Let $\{1_G\} \trianglelefteq H_1 \cdots \trianglelefteq H_n = G$ and $\{1_G\} \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_r = G$ be two composition series. We want to show that $\{K_{i+1}/K_i\}$ and $\{H_i + 1/H_i\}$ are isomorphic up to reordering. We proceed by induction on the length of the shorter of the two series. With out loss of generality we can assume that $\{H_i\}$ is the shorter of the two sequences.
The base case is where the shorter series has length 1. This would mean that the group is trivial, and thus only admits on composition series.
Now suppose that the result is true when one of the sequences is of length less that $n$, and consider the composition series $\{H_i\}_{i=0}^n$ and $\{K_i\}_{i=0}^r$. The first case is when $H_{n-1} = K_{r-1}$. In this case the series up to this point will both by isomorphic by the inductive hypothesis. Since the two groups are isomorphic, and obviously the series after this point are identical.
Now suppose that $H = H_{n-1}$ and $K = K_{r-1}$ and $H \neq K$. Let $L = H \cap K$. Since $H$ and $K$ are normal in $G$, $L$ is normal in $G$ as well, and thus $L$ is normal in both $H$ and $K$.
Now we claim that $L$ has a composition series given by $L_i = L \cap H_i$. The proof of this claim breaks into the following steps:

1. $L_0 = \{1_G\}$ and $L_{n-1} = L$.
2. $L_i \trianglelefteq L_{i+1}$ because $H_i \trianglelefteq H_{i+1}$.
3. $L_{i+1}/L_i$ is simple: For all $i$, note that $L_i \trianglelefteq H_i$ FINISH ME

Now we have the composition series $L_0 \trianglelefteq L_1 \trianglelefteq \cdots \trianglelefteq L$ and we could potentially extend it by adding either $H$ or $K$ to the end. To do this we would have to show that $H/L$ and $K/L$ are simple. Since $H$ and $K$ are normal in $G$, we know that the subgroup that is their product, $HK$ is normal in $G$ as well by the lemma from above. Then by the correspondence theorem, $HK/K \trianglelefteq G/K$. We know that $G/K$ is simple, so $HK/K$ must either be all of $G/K$ or the trivial subgroup. If it were the trivial subgroup that would mean $H = K$ which we handled in the previous case. Thus we assume that $HK = G$. By the second isomorphism theorem, we have that

$$H/L = H/(H \cap L) \cong HK/K = G/K.$$

Since $G/K$ is simple, so is $H/K$. Similar arguments work to show that $K/L$ is simple. Now we have two compositions series, $L_0 \trianglelefteq \cdots \trianglelefteq L_{n-1} \trianglelefteq H$ and $H_0 \trianglelefteq \cdots \trianglelefteq H$. By the inductive hypothesis these are isomorphism. Similarly, we have $L_0 \trianglelefteq \cdots \trianglelefteq L_{n-1} \trianglelefteq K$ and $K_0 \trianglelefteq \cdots \trianglelefteq K$ which must be isomorphic. This specifically means that $n-1 = r-1$, so $r = n$ ($r$ is the length of the composition series $\{K_i\}$ and $n$ is the length of the composition series $\{H_i\}$.
Now the sequences that end in $H$ and in $K$ have identical (even in order) factor groups except for the last ones. This means that $\{H_i\}_{i=0}^{n-1}$ and $\{K_i\}_{i=0}^{n-1}$ are isomorphic with the exception of the last factor groups. These last factor groups are $H/L = H/(H \cap K) \cong HK/K = G/K$, and $K/L = K/(K \cap H) \cong KH/H = G/H$, respectively. We resolve this issue by adding $G$ to the end of both of the series, which will add a final factor group of $G/H$ and $G/K$, respectively. This shows that $\{H_i\}$ is isomorphic to $\{K_i\}$.
Now we can add on either $H$ and then $G$ or $K$ and then $G$ to the end of the composition series $\{L_i\}$. This would add the factors $G/H, H/L = H/(H \cap K) \cong HK/K = G/K$ or $G/K, K/L = K/(K \cap H) \cong KH/H = G/H$. □

**Theorem 2.0.10**
If $G$ has a composition (or principal) series, and $N \trianglelefteq G$ then there is a composition (or principal) series containing $N$.

*Proof.* Start with a composition (or principal) series, and $\{1_G\} \trianglelefteq N \trianglelefteq G$. By Schrier's Theorem, they admit isomorphic refinements, but we obviously can not refine the composition (or principal) series any more. Thus the refinement of $\{1_g\} \trianglelefteq N \trianglelefteq G$ will have all simple factor groups, and will be a composition (or principal) series containing $N$. □

**Definition 2.0.11**
A group $G$ is solvable if it has a composition series with Abelian factor groups.

We don't get to see the applications of solvable groups, but apparently they exist in Galois Theory.

**Example 2.0.12**
- $\{1\} \trianglelefteq A_3 \trianglelefteq S_3$ is a composition series with factor groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (up to isomorphism), so $S_3$ is solvable.
- $\{1\} \trianglelefteq A_5 \trianglelefteq S_5$ is a composition series with non-abelian factor groups. Since all composition series are isomorphic, no composition series of $S_5$ has all abelian factor groups. Thus $S_5$ is not solvable.

**Definition 2.0.13**
If $G$ is a group then the center of $G$ is $Z(G) = \{z \in G \mid zx = xz \text{ for all } x \in G\}$.

Clearly the center is a normal subgroup. Using the correspondence theorem, we can consider $Z_1$, the subgroup of $G$ corresponding to $Z(G/Z(G)) \trianglelefteq G/Z(G)$, $Z_2$ the subgroup of $G$ corresponding to $Z(G/Z_1(G))$, and so on. Then we get the *ascending central series of* $G$:

$$\{1\} \trianglelefteq Z(G) \trianglelefteq Z_1(G) \trianglelefteq \cdots .$$

**Example 2.0.14**

- $Z(S_3) = \{1\}$. This means that the ascending central series is just $\{1\} \trianglelefteq \{1\} \trianglelefteq \{1\} \trianglelefteq \cdots$ .
- $Z(D_4) = \langle r^2 \rangle$. Then $D_4/Z(D_4)$ has 4 elements, and all groups with 4 elements are abelian, meaning that the center is the whole group. Thus we get the ascending central series $\{1\} \trianglelefteq \langle r^2 \rangle \trianglelefteq D_4 \trianglelefteq D_4 \trianglelefteq \cdots$ .

# 3 The Sylow Theorems

**Definition 3.0.1**

If $G$ is a group and $X$ is a set, then $G$ acts on $X$ if there is a map $\phi : G \times X \to X$ where $\phi(g, x)$ is denoted $g \cdot x$ such that

1. $1_G \cdot x = x$ for all $x \in X$, and
2. $h \cdot (g \cdot x) = (hg) \cdot x$.

The *orbit* of a group element $x$ under the action of $G$ is $\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\}$ and the stabilizer of $x$ is $\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$. Clearly the stabilizer is a normal subgroup of $G$.

**Theorem 3.0.2**

If a finite group $G$ acts on a set $X$, then for any $x \in X$ there is a bijection $\phi : G/\mathrm{Stab}_G(x) \to \mathrm{Orb}_G(x)$. In particular, $|G|/|\mathrm{Stab}_G(x)| = |\mathrm{Orb}_G(x)|$. Moreover, the orbits for a partition of $X$.

**Definition 3.0.3**

The fixed points of a group action of $G$ on $X$ is the set $X^G = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$.

Remarks:

- $x \in X^G \iff \mathrm{Orb}_G(x) = \{x\}$, and
- If $|X| < \infty$, then $|X| = |X^H| + \sum_{i=0}^{m} |O_i|$, where the sum is over the orbits of $X$ that have more than one element.

**Lemma 3.0.4**

Let $G$ be a group of order $p^n$ where $p$ is some prime and $n$ is sime positive integer, and let $G$ act on some finite set $X$. Then $|X| \equiv |X^G| \bmod p$.

*Proof.* By the above theorem, $|\mathrm{Orb}_G(x)| \mid |G|$, which means that $|\mathrm{Orb}_G(x)|$ is a power of $p$. Then we can apply the second remark to get the result. $\square$