Pointwise Convergence Versus Convergence in L^p

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Introduction

We have learned about two different types of convergence for sequences of functions in L^p. One is the pointwise limit, and the other is the limit with respect to the L^p-norm. However, we have seen that these two forms of convergence are distinct.

Example 1. See figure 1. Consider the sequence of functions $\{f_n\}$ in $L^1[0,1]$ where for each n,

$$f_n(x) = \begin{cases} 2n^2x & 0 \leq x \leq \frac{1}{2n} \\ -2n^2(x - \frac{1}{n}) & \frac{1}{2n} < x \leq \frac{1}{n} \\ 0 & otherwise \end{cases}$$

Note that this is a triangle with a base of length $1/n$ and a height of n. This sequence of functions converges pointwise to the function $f(x) = 0$ as n goes to ∞. However, $lim_{n\to\infty}||f_n - 0||_1 = lim_{n\to\infty} \int_0^1 |f_n| = lim_{n\to\infty}(\frac{1}{2}) \neq 0$, so the function does not converge to 0 with respect to the L_1-norm.

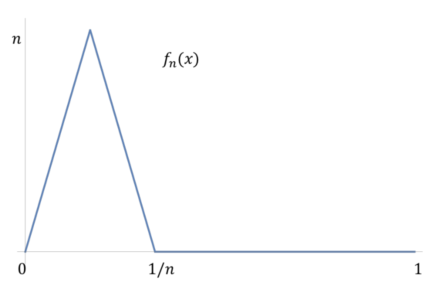

Figure 1: Generalized plot of $f_n(x)$ for Example 1.

Example 2. See figure 2. A sequence of functions may converge with respect to the norm but not pointwise. Consider the following sequence $\{f_n\}$ in $L^1[0,1]$ where $f_1 = \chi_{[0,\frac{1}{2}]}, f_2 = \chi_{[\frac{1}{2},1]}, f_3 = \chi_{[0,\frac{1}{4}]}, f_4 = \chi_{[\frac{1}{4},\frac{1}{2}]}, ..., f_7 = \chi_{[0,\frac{1}{8}]}, f_8 = \chi_{[\frac{1}{8},\frac{1}{4}]}, ...$ Then for any $N \in \mathbb{R}$ and $x \in [0,1]$, there will always be some $n > N$

for which $f_n(x) = 1$. Thus, the sequence does not converge pointwise anywhere in $[0, 1]$. However, $lim_{n\to\infty} ||f_n - 0||_1 = lim_{n\to\infty} \int_0^1 f_n = lim_{k\to\infty} \frac{1}{2^k} = 0$, so the sequence converges to 0 with respect to the norm.

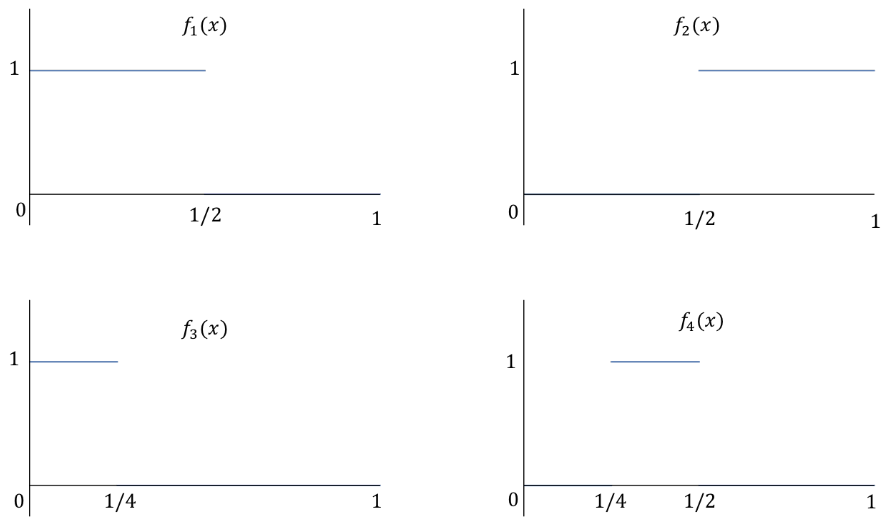

Figure 2: Plots of $f_n(x)$ for $n = 1, 2, 3, 4$ from Example 2.

Our goal is to find a condition for a sequence of functions in L^p that ensures that the limit is the same both point-wise and with respect to the norm.

Preliminaries and Notation

Definition 3. *Let $\{f_n\}$ be a sequence of functions. We say that this sequence is Cauchy if, for all $\epsilon > 0$, there exists an N in $\mathbb{N}$ such that for all $n > N$,*

$$|f_{n+k} - f_n| < \epsilon$$

for all k in $\mathbb{N}$.

In what follows, X will always refer to a normed vector space with norm $|| \cdot ||$.

Lemma 4 (Borel-Cantelli Lemma)**.** *Let E_k be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $\sum_{k=1}^{\infty} m(E_k)$ converges. Then almost all x in $\mathbb{R}$ belong to at most finitely many of the E_k's.*

Proof. Assume that E_k is a set of measurable sets such that $\sum_{k=1}^{\infty} m(E_k)$ converges. We know that

$$m(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = 0$$

Now let A be the set of x that are in infinitely many E_k's. Then for any n, there exists a $k \geq n$ such that x is in E_k. So x is in $\bigcup_{k=n}^{\infty} E_k$ for all n. Thus

$$x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n} E_k$$

which means that

$$m(A) \leq m(\bigcap_{n=1}^{\infty} \bigcup_{k=n} E_k) = 0$$

Thus

$$m(A) = 0$$

This shows that the set of points that are in infinitely many E_k's has measure 0, which means that almost all x in $\mathbb{R}$ belong to at most finitely many of the E_k's □

Condition for Convergence

Definition 5. *A sequence is* ***rapidly Cauchy*** *if there is a convergent positive series* $\sum_{k=1}^{\infty} \epsilon_k$ *where*

$$||f_{k+1} - f_k|| \leq \epsilon_k^2 \text{ for all } k.$$

Observation: Suppose that $\{f_n\}$ is a sequence in X, and $\{a_k\}$ is a sequence of non-negative numbers such that $||f_{k+1} - f_k|| \leq a_k$. Then for all k and n, $f_{n+k} - f_n = \displaystyle\sum_{j=n}^{n+k+1} [f_{j+1} - f_j]$. The triangle inequality for norms implies that

$$||f_{n+k} - f_n|| \leq \sum_{j=n}^{n+k+1} ||f_{j+1} - f_j|| \leq \sum_{j=n}^{n+k+1} a_j \leq \sum_{j=n}^{\infty} a_j \tag{1}$$

Proposition 6. *Every rapidly Cauchy sequence in X is Cauchy (with respect to the norm).*

Proof. Let $\{f_n\}$ be a rapidly Cauchy sequence in X, then there is a convergent series of non negative integers $\sum_{k=1}^{\infty} \epsilon_k$ with the property that $||f_{k+1} - f_k|| \leq \epsilon_k^2$ for all k. Then by Equation 1, we see that

$$||f_{n+k} - f_n|| \leq \sum_{k=n}^{\infty} \epsilon_k^2 \tag{2}$$

Since the summation $\sum_{k=1}^{\infty} \epsilon_k^2$ converges, for any $\epsilon > 0$, there is an N such that if $n \geq N$ then $\sum_{k=n}^{\infty} \epsilon_k^2 < \epsilon$. This along with equation 2 implies that the sequence is Cauchy. □

Proposition 7. *Every Cauchy sequence has a rapidly Cauchy sub-sequence.*

Proof. Assume that $\{f_n\}$ is a Cauchy sequence. Then we know that for any k, there is some n_k such that $||f_{n_k+1} - f_{n_k}||_p < (1/2)^k$. Since the series $\sum_{k=1}^{\infty}(1/2)^k$ converges, the sub-sequence f_{n_k} is rapidly Cauchy. $\square$

Theorem 8. *Let E be a measurable set and let $1 \leq p < \infty$. Then every rapidly Cauchy sequence in L^p converges with respect to the L^p-norm and point-wise almost everywhere to a function in L^p.*

Proof. Assume that $\{f_n\}$ is a rapidly Cauchy sequence in L^p. Then we know that for all n, f_n takes real values almost everywhere. We know that there is some sequence of real positive number $\{\epsilon_k\}$ such that the series $\sum_{k=1}^{\infty}\epsilon_k$ converges and

$$||f_{k+1} - f_k||_p \leq \epsilon_k^2 \text{ for all } k. \tag{3}$$

By raising both sides to the p, we get that

$$\int_E |f_{k+1} - f_k|^p \leq \epsilon_k^{2p}. \tag{4}$$

Note that $|f_{k+1} - f_k| \geq \epsilon_k$ if and only if $|f_{k+1} - f_k|^p \geq \epsilon_k^p$. If $M_k = \{x \in E : |f_{k+1}(x) - f_k(x) \geq \epsilon_k\}$, then $m(M_k) = m\{x \in E : |f_{k+1} - f_k|^p \geq \epsilon_k^p\}$. Using this along with Chebychev's Inequality and equation 4, we get that

$$m(M_k) \leq \frac{1}{\epsilon_k^p}\int_E |f_{k+1} - f_k|^p \leq \epsilon_k^p. \tag{5}$$

Since $\sum_{k=1}^{\infty}\epsilon_k$ converges, $\epsilon_k \to 0$ as $k \to \infty$. Thus there is some N such that if $n \geq N$ then $\epsilon_n < 1$. For such n, $\epsilon_n^p \leq \epsilon_n$. Since $\epsilon_n < \epsilon_k^p$ for only finitely many values, $\sum_{k=1}^{\infty}\epsilon_k^p$ must converge as well. This in turn implies that $\sum_{k=1}^{\infty} m(M_k)$ converges.

So we can apply the Borel-Cantelli Lemma to find that $m(M_0) = 0$ where $M_0 := \{x \in E : x \text{ is in infinitely many } M_k\}$. Then for all $x \notin M_0$, there is some $K(x) \in \mathbb{N}$ such that for all $k \geq K(x)$, $|f_{k+1}(x) - f_k(x)| < \epsilon_k$. To prove this, assume to the contrary that for some $x \notin M_0$, for all K there exists some $k \geq K$ such that $|f_{k+1}(x) - f_k(x)| \geq \epsilon_k$. Then there must be infinitely many k_i such that $|f_{k_1+1}(x) - f_{k_i}(x)| \geq \epsilon_k$. Then $x \in M_{k_i}$ for infinitely many k_i, but this contradicts the assumption that $x \notin M_0$.

By the triangle inequality for norms, for all $n \geq K(x)$ and k,

$$\begin{aligned}
|f_{n+k}(x) - f_n(x)| &\leq \sum_{j=n}^{n+k-1} |f_{j+1}(x) - f_j(x)| \\
&\leq \sum_{j=0}^{n+k-1} \epsilon_j \\
&\leq \sum_{j=0}^{\infty} \epsilon_j.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} \epsilon_k$ converges, we know that $\sum_{k=n}^{\infty} \epsilon_k \to 0$ as $k \to \infty$. This means that for any $\epsilon > 0$ we can find an N such that if $n \geq N$, $|f_{n+k}(x) - f_n(x)| \leq \sum_{j=n}^{\infty} \epsilon_j \leq \epsilon$, which proves that $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the limit of $f_k(x)$ exists, so we denote it by $f(x)$. Then f_k converges pointwise almost everywhere to f (it only does not converge for $x \in M_0$ and $m(M_0) = 0$).

We have found the pointwise limit of $\{f_n\}$, and now what remains to show is that this is also the limit with respect to the L^p norm. To do this we need to show that $||f - f_n||_p \to 0$ as $n \to \infty$.

By equations 1 and 4, we get that

$$\int_E |f_{n+k} - f_n|^p \leq \left[\sum_{j=n}^{\infty} \epsilon_j^2\right]^p \text{ for all } n, k.$$

Since the sequence $\{|f_{n+k} - f_n|^p\}_{k=1}^{\infty}$ is a sequence of non-negative functions in $\mathcal{L}[E]$, and the pointwise limit of this sequence as $k \to \infty$ is $|f - f_n|^p$, we can apply Fatou's Lemma to find that

$$\begin{aligned} \int_E |f - f_n|^p &\leq \lim_{k\to\infty} \int_E |f_{n+k} - f_n|^p \\ &\leq \left[\sum_{j=n}^{\infty} \epsilon_j^2\right]^p. \end{aligned}$$

This means that

$$||f - f_n||_p \leq \sum_{j=n}^{\infty} \epsilon_j^2.$$

Since $\sum_{j=n}^{\infty} \epsilon_j^2 \to 0$ as $n \to \infty$, this implies that f is the limit point of the sequence (with repect to the L^p-norm) $\{f_n\}$, and since L$^p[E]$ is complete, this meas that $f \in$ L$^p[E]$. Thus f is both the limit with respect to the norm and point-wise almost everywhere, as desired. □

Example 9. Recall from Example 2 the sequence of functions $\{f_n\}$ that converges with respect to the L^1-norm but not pointwise. Proposition 7 and Theorem 8 tells us that $\{f_n\}$ has a rapidly cauchy subsequence and thus converges to the same function f pointwise and with respect to the norm. We have the subsequence $\{g_n\}$ with $g_1 = \chi_{[0,\frac{1}{2}]}, g_2 = \chi_{[0,\frac{1}{4}]}, g_3 = \chi_{[0,\frac{1}{8}]} \dots$ and in general $g_n = \frac{1}{2^n}$ Since $\{f_n\}$ converges with respect to the L^1-norm to $f = 0$, the subsequence $\{g_n\}$ must also converge to $f = 0$ with respect to the norm. Furthermore $\{g_n\}$ is rapidly Cauchy since for each n, $||g_n||_1 = 1/2^n$. By Theorem 8, then, we should expect $\{g_n\}$ to converge pointwise to $g = 0$, and indeed it does since for any x, $g_n(x)$ eventually is 0.

Theorem 10 (Riezs Fischer)**.** *Let E be a measurable set and $1 \leq p < \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $f_n \to f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f.*

Proof. Let $\{f_n\}$ be a Cauchy sequence of functions in $\mathrm{L}^p(E)$. Since $\{f_n\}$ is Cauchy, then we know by Proposition 7 that there exists a rapidly Cauchy subsequence $\{f_{n_k}\}$. We know from Theorem 8 that $\{f_{n_k}\}$ converges both with respect to the $\mathrm{L}^p(E)$ norm and pointwise a.e. on E to a function in $\mathrm{L}^p(E)$. We also know that a Cauchy sequence in a Vector space converges if it has a convergent subsequence, which means that $\{f_n\}$ converges to f with respect to the $\mathrm{L}^p(E)$ norm. $\square$

Our Own Exploration

In our exploration, we found examples of sequences which converge pointwise but not with respect to the norm (Example 1) and vice versa (Example 2). This led to the question: Can a function converge both pointwise and with respect to the norm, but to different functions?

Theorem 11 (Buck-Nadiga-Soufan)**.** *Let f_n be a sequence that converges in L^p to f and pointwise to g. Then $f = g$.*

Proof. Let $\{f_n\}$ be such a sequence, converging in L^p to f and pointwise to g. Since f_n converges in L^p, $\{f_n\}$ is Cauchy and thus by Proposition 7 has a rapidly Cauchy subsequence $\{f_{n_k}\}$. This subsequence must also converge to f with respect to the norm. Then by Theorem 7, $\{f_{n_k}\}$ also converges pointwise to f. But since $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$, its pointwise limit must also be g since the pointwise limit of $\{f_n\}$ is g. $\{f_{n_k}\}$ can only have a single pointwise limit, so it must be that $f = g$. $\square$

Example 12. Recall Example 1. We were able to show that $\{f_n\}$ converges pointwise to 0 but does not converge to 0 with respect to the L^1-norm. Now that we have Theorem 11, we can be sure that the limit of $\{f_n\}$ with respect to the norm must not exist at all because if it did exist it would have to be 0.