Operator Algebras

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0 Introduction

These are the notes based on the Operator Algebras course taught in Fall of 2019 by Dr Joachim Zacharias. There was no single text for this course.

If you find any errors please tell me (or email me at asnadiga@gmail.com).

0.1 Notation

In this course all spaces that we consider will be normed linear spaces over $\mathbb{C}$.

0.2 Background

Definition 0.1
A norm on a space X is a function $||\cdot|| : X \to [0, \infty)$ such that for all $x, y \in X$ and $\lambda \in \mathbb{C}$

1. $||x + y|| \leq |||x|| + ||y||$,
2. $||\lambda x|| = |\lambda| \cdot ||x||$
3. $||x|| = 0$ if and only if $x = 0$.

Definition 0.2
A normed space X is a Banach space if it is complete with respect to its norm (meaning that every sequence that is Cauchy with respect to the norm converges with respect to the norm to some value in X).

Example 0.3
The following are Banach spaces:

1. Any finite dimensional normed linear space.
2. $\ell^\infty = \{(a_n) \in \mathbb{C}^\mathbb{N} \mid (a_n) \text{ bounded}\}$ with norm $||(a_n)|| = \sup\{|a_i| \mid i \in \mathbb{N}\}$.
3. $C(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ is continuous}\}$, with the norm $||f||_\infty = \sup\{|f(t)| \mid t \in \Omega\}$, where Ω is a compact topological space.
4. For X and Y, consider $B(X, Y) = \{T : X \to Y \mid ||T|| < \infty\}$ where $||T|| = \sup\{||Tx|| \mid x \in X, ||x|| < 1\}$. This is called the set of bounded linear operators from X to Y.
5. $B(X, \mathbb{C})$ is called the dual of X.

Definition 0.4
Inner product spaces carry the norm $||x|| = \langle x, x\rangle^{1/2}$. Any inner product space that is complete with respect to this norm is called a *Hilbert space*.

Clearly all Hilbert spaces are Banach spaces as well.

Example 0.5
The following are Hilbert spaces:

1. $\mathbb{C}^n$ with the standard inner product.

2. $\ell^2 = \{(a_n) \in \mathbb{C}^{\mathbb{N}} \mid \sum |a_i|^2 < \infty$.

Definition 0.6
An *algebra* is a linear space X with an associative bilinear multiplication, meaning that:
1. $(x + \lambda y)z = xz + \lambda yz$,
2. x $x(y + \lambda z) = xy + \lambda xz$, and
3. (xy)z=x(yz)

for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$.

An algebra is a normed algebra is X is normed, and is a Banach algebra is X is a Banach space.

Definition 0.7
A *unit* in an algebra A is a unique element, denoted 1_A such that $1_A x = x 1_A = x$ for all $x \in A$. If A has a unit then A is a *unital algebra*.

Example 0.8
The following are algebras:
1. $C(\Omega)$ with pointwise multiplication is a unital commutative algebra (when Ω is compact.
2. $B(X) = B(X, X)$ with multiplication given by composition is a unital normed algebra. $B(X)$ is a Banach algebra if and only if X is a Banach space. $B(X)$ is not commutative if and only if $\mathrm{Dim}(X) > 1$. Every Banach Algebra is contained in some $B(X)$.
3. (2) when X is a Hilbert space.

Examples (1) and (3) also carry an involution.

Definition 0.9
An involution on an algebra A is a map $* : A \to A$ such that for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$,
1. * is antilinear: $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$,
2. * is antimultiplicative: $(xy)^* = y^* x^*$, and
3. * is involutive: $(x^*)^* = x$.

On $C(\Omega)$ the involution is given by $f^*(t) = \overline{f(t)}$. On $B(H)$ the involution of T is the adjoint of T, which is the unique operator satisfying that $\langle T^* x, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$.

Definition 0.10
A Banach algebra is a C*-algebra if it carries an invultion * such that $||xx^*|| = ||x||^2$.

Both $C(\Omega)$ and $B(H)$ are C*-algebras.

1 Banach Algebras

Example 1.1
Given a Banach algebra A, define the Banach Algebra $\overline{A} = A \oplus \mathbb{C}$ with addition being component wise, multiplication such that $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$, and the norm $||(a, \lambda)|| = ||a|| + |\lambda|$. Then $\overline{A}$ is a unital Banach algebra with unit $(0, 1)$. An involution on A extends to $\overline{A}$ through $(a, \lambda)^* = (a^*, \overline{\lambda})$.

Example 1.2
Let G be a discrete group. Then $\ell^1(G) = \{f : G \to \mathbb{C} \mid \sum_{g\in G} |f(g) < \infty\}$ with norm $||f||_1 = \sum_{g\in G} |f(g)|$ is a Banach space. Then if we defined multiplication through $(fg)(s) = \sum_{g\in G} f(t)g(t^{-1}s)$ we have a Banach Algebra.

Definition 1.3
Given an algebra A, we denote the set of invertible elements of A by $G(A)$

Proposition 1.4
Let A be a Banach algebra with unit 1_A. Then the following are true.

1. For $x \in A$ such that $||x|| < 1$, $1 - x$ is invertible. More generally if $||x|| < \lambda$ then $\lambda - x$ is invertible.
2. $G(A)$ is open and the inverse map is continuous.
3. $G(A)$ is a topological group, meaning that the inverse and multiplication functions are continuous.

Proof.

1. We can construct the inverse of $1 - x$, $\sum_{n=0}^{\infty} x^n$. Then $(1 - x) \sum_{n]=0}^{n=1_A}$.
2. Let $x \in G(A)$ to show that $G(A)$ is open, we will show that the open ball of radius $||x^{-1}||^{-1}$ is contained in $G(A)$. So suppose that $y \in A$ such that $||x - y|| < ||x^{-1}||^{-1}$. Then we can write $y = x - (x - y) = x(1 - x^{-1}(x - y))$. Then since $||x^{-1}(x - y)|| \leq ||x^{-1}|| \cdot ||x - y|| < 1$, the element $1 - x^{-1}(x - y)$ is invertible. Since x is also invertible $y = x(1 - x^{-1}(x - y))$ is invertible. By applying the formula for the inverse from part (1), to $1 - x^{-1}(x - y)$ we can show continuity.
3. We already know that multiplication is continuous in A, and we have shown that inversese are continuous.

□

Definition 1.5
A continuous path in a topological space Ω connecting $\omega_0 \in \Omega$ to $\omega_1 \in \Omega$ is a continuous mapping $p : [0, 1] \to \Omega$ such that $p(0) = \omega_0$ and $p(1) = \omega_1$. A topogical space Ω is path connected if any two point can be joined by a continuous path.

Definition 1.6
$G_0(A)$ is the set of elements in $G(A)$ that can be connected to 1_A by a path in $G(A)$. More formally, for all $x \in G_0(A)$, there is a continuous $p : [0, 1] \to G(A)$ such that

$p(0) = x$ and $p(1) = 1_A$.

Proposition 1.7
$G_0(A)$ is
1. connected,
2. open, and
3. normal in $G(A)$.

Proof.
1. Since every element of $G_0(A)$ is path connected to 1_A, we can construct a path between any two points.
2. We will show that for any $x \in G_0(A)$, the open ball of radius $||x^{-1}||^{-1}$ about x is contained in $G_0(A)$. Let $p(t) = x + th$. This is a path connecting x to $x + h$. Since $G_0(A)$ is path connected, it will suffice to show that this entire path is contained in $G(A)$, because then we can construct a path from $x + h$ to 1_A that stays in $G(A)$. Note that for any t, $p(t) = x + th = x(1 - (-tx^{-1}h)$. $|| - tx^{-1}h|| \leq |t| \cdot ||x^{-1}|| \cdot ||h|| < 1 \cdot ||x^{-1}|| \cdot ||x^{-1}||^{-1} = 1$. Thus by the previous proposition, $p(t)$ is invertible.
3. Let $x \in G_0(A)$. We want to show that for any $a \in G(A)$, $axa^{-1} \in G_0(A)$. $x \in G_0(A)$ implies that there is some path p connecting x to 1_A. Then $ap(t)a^{-1}$ is a path connecting axa^{-1} to 1_A that stays in $G(A)$, meaning that $axa^{-1} \in G_0(A)$.

□

Proposition 1.8
An open subgroup of a topological group is closed.

Proof. Let G be a topological group and let $H < G$ be an open subgroup. Then if the (possibly finite) set $\{g_0 = 1_G, g_1, ...\}$ is a set of unique coset representatives of H in G, then we have that $G = \cup g_i H$. We also know that since multiplication is continuous in a topological group, H open implies that $g_i H$ is open as well. Thus $G \setminus H = \cup_{i \neq 0} g_i H$ is open, meaning that H is closed. □

This proposition implies that $G(A)/G_0(A)$ is always a discrete group.

Definition 1.9
Let A be an algebra with unit 1. Then for each $x \in A$ we defined the *spectrum* of x as

$$\sigma(x) = \{\lambda \in \mathbb{C} \mid \lambda - x \notin G(A)\}.$$

Proposition 1.10
Let A be a Banach algebra then for any $x \in B$ $\sigma(x) \neq \emptyset$

Proof. This has something to do with holomorphic functions and stuff. I really cant work with that. □

Proposition 1.11
Let A be a Banach algebra with a unit element. Then for every $x \in A$, the following are true:

1. $\sigma(x)$ is closed, and
2. $\sigma(x) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| < ||x||\}$.

Proof.

1. The complement of $\sigma(x)$, $C \setminus \sigma(x)$ is the preimage of $G(A)$ under the map $\lambda \mapsto \lambda - x$. This map is continuous and $G(A)$ is open so this preimage is open. This $\sigma(x)$ is closed.
2. Let $\lambda \in \sigma(x)$, and suppose for contradiction that $|\lambda| > ||x||$. Then we know that $\lambda - x$ is invertible which contradicts the assumption that $\lambda \in \sigma(x)$

□

Proposition 1.12
Let A be a unital algebra. If $x, y \in A$, then $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.

Proof. Let $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$. We want to show that $\lambda - xy \in G(A) \iff \lambda - yx \in G(A)$. By symmetry we only need to prove one direction. We can also assume with out loss of generality that $\lambda = 1$ because if it is not we can scale everything that follows appropriately.
$\lambda - xy \in G(A)$ means that $1 - xy$ is invertible. This means that $-(1 - xy) = (xy - 1)$ is invertible, so there exists some $u \in A$ such that $u(xy - 1) = (xy - 1)u = 1_A$. Let $w = 1 - yux$. Then we have that

$$(1 - yx)w = 1 - yx - yux + yxyux = 1 - yx - y(1 - xy)ux = 1 - yx - (-y1_Ax) = 1_A$$

and

$$w(1 - yx) = 1 - yx - yux + yuxyx = 1 - yx - yu(1 - xy)x = 1 - yx - (-y1_Ax) = 1_A$$

. Thus w is the inverse of $(1 - yx)$; $1 - yx \in G(A)$. □

Proposition 1.13
Let B be a Banach algebra with unit 1_B, and let $A \subset B$ be a closed subalgebra such that $1_B \in A$. Then for all $x \in A$, the following are true:

1. $\sigma_B(x) \subseteq \sigma_A(x)$, and
2. $\delta(\sigma_A(x)) \subseteq \delta(\sigma_B(s))$.

Proof.

1. Since $G(A) \subseteq G(B)$, there may be inverses in B that are not in A. Thus there may be values of λ that are in $\sigma_A(x)$ that may not be in in $\sigma_B(x)$. On the other hand if an element is not invertible in B then it is not in A either.
2. Let $\lambda \in \delta(\sigma_A(x))$. We want to show that $\lambda \in \delta(\sigma_B(x))$. Since λ is a boundary point point there is a sequence (λ_n) in $\sigma_A(x)^C$ that converges to λ. Now assume for contradiction that $\lambda \notin \delta(\sigma_B(x))$. Then $\lambda \notin \sigma_B(x)$, meaning that $(\lambda - x)^{-1}$ exists in B. Since addition and inverses are continuous, the sequence $(\lambda_n 1_B - x)^{-1}$ converges to $(\lambda 1_B - x)^{-1}$. Since $\lambda_n \notin \sigma_A(x)$ for every n, the element $(\lambda_n - x)^{-1}$ exists in A for ever n. Since A is closed, the limit of this sequence, $(\lambda - x)^{-1}$ is in A. This contradicts the assumption that $\lambda \in \sigma_A(x)$.

□

This means that in the situation described above, if $\sigma_A(x) \subseteq \mathbb{R}$, then $\sigma_A(x) = \sigma_B(x)$.

Definition 1.14
Let A be a Banach algebra with unit 1_A. Then for $x \in A$, the *spectral radius* of x is defined as

$$r(x) := \max\{|\lambda| \mid \lambda \in \sigma(x)\}.$$

Since $\sigma(x)$ is closed and bounded, $\{|\lambda| \mid \lambda \in \sigma(x)\}$ is closed in bounded in $\mathbb{R}$, meaning that the maximum of this set is well defined.

Theorem 1.15
Let A be a Banach algebra with unit 1_A, then for all $x \in A$, $r(x) = \lim_n ||x^n||^{1/n}$.

Proof. This also has to do with holomorphic functions and I dont understand it. □

Theorem 1.16
Let A be a Banach algebra in which ever non zero element is invertible. Then $A \approx \mathbb{C}$.

Proof. Let $x \in A$, we know that $\sigma(a) \neq \emptyset$, so let $\lambda \in \sigma(a)$, meaning that $\lambda - x$ is not invertible. Since all of the non-zero elements of A are invertible, $\lambda - x = 0$, which means that every $x \in A$ is a scalar multiple of the identity element, which gives an obvious isomorphism to $\mathbb{C}$. □

2 Ideals and Quotients

Definition 2.1
Given a Banach algebra A, a subalgebra (closed under addition and multiplication) I is an ideal if $AI \subseteq I$ and $IA \subseteq I$.

An algebra is *simple* if the only ideals are the full algebra and the empty set.

We can also note that if A is a unital algebra then any non-trivial ideal can not contain 1_A (because then $A \subseteq AI$). More generally, any non-trivial ideal can not contain any invertible elements.

Definition 2.2
If A is an algebra with I an ideal, then we can form the *quotient algebra* $A/I = \{a + I \mid a \in A\}$. The operations are as follows: $(a + I) + (b + I) = (a + b) + I$, and $(a + I)(b + I) = ab + I$.

Proposition 2.3
If A is a Banach algebra and $I \trianglelefteq A$ is an ideal then the following are true.

1. The closure of I, $\overline{I}$ is also an ideal.
2. $||1_A + z|| \geq 1$ for all $z \in I$ (unless $A = I$).

3. Further suppose that A is unital. If I is a proper ideal of A, then $\overline{I}$ is a proper ideal of A.

Proof. Let A be a unital Banach algebra, and let $I \trianglelefteq A$.

1. Let $x \in \overline{I}$. Then there is some sequence (x_n) in I that converges to x. Since I is an ideal, for any $a \in A$, the sequence (ax_n) is in I, and by the continuity of multiplication, this sequence converges to ax. Thus $ax \in I$ for any $x \in I$ and $a \in A$.
2. Suppose that for some $z \in I$, $||1_A + z|| < 1$. Then we know that $q - (1_A + z) = -z$ is invertible in A, which means that z is invertible as well. This means that $I = A$.
3. Suppose that $I \neq A$, but $\overline{I} = A$. Then we know that $-1_A \in \overline{I}$, so there is a sequence (x_n) in I that converges to -1_A. This means that the sequence $(1 + x_n)$ converges to 0_A. By (2), we know that $||1_A + x_n|| \geq 1$, but this makes it impossible for $(1_A + x_n)$ to converge to 0_A. Thus we conclude that $\overline{I} \neq A$.

□

2.1 Maximal Ideals

Definition 2.4
Given an algebra A and an ideal $I \trianglelefteq A$, we say that I is a *maximal ideal* if $J \trianglelefteq A$ is proper ideal that contains I implies that $J = I$

Proposition 2.5
Any maximal ideal of a (i think unital) Banach algebra is closed.

Proof. Let A be a Banach algebra and $I \trianglelefteq A$ be a maximal ideal. Then we know that $\overline{I}$ is also a proper ideal, and it contains I. Thus $\overline{I} = I$. □

Proposition 2.6
If A is a Banach algebra, then $I \trianglelefteq A$ is a maximal ideal if and only if A/I is simple.

Proof. We will show a correspondence between ideals of A and ideals of A/I, which will be used to prove the proposition. First start with an ideal of A, J, then we can construct the ideal of A/I, $K = \{j + I \mid j \in J\}$. Similarly, given an ideal of A/I, K, we can construct the ideal $J = \cup_{a+I \in K} a + I$. (see correspondence theorem for rings).
Now suppose that I is a maximal ideal in A. Then suppose that K is an ideal in A/I. Then the corresponding ideal J clearly contains I. This means that J is either I or A. If $J = I$ then K is the empty ideal, and if $J = A$, then $K = A/I$. Thus A/I is simple.
Now suppose that A/I is simple. Then let J be any ideal that contains I. The corresponding ideal K in A/I must be either the empty ideal or all of A/I. This would imply that J is either I or A, which means that I is maximal. □

Proposition 2.7
If A is an algebra then every proper ideal of A is contained in a maximal subideal of A.

Proof. The set of proper ideals is a partially ordered set with comparison given my inclusion. We also know that for any fully ordered subset, there is a maximal element. Thus we can apply Zorn's lemma to prove the proposition. □

Proposition 2.8
Let A be a unital Banach algebra. Then if $I \trianglelefteq A$ is a maximal ideal, $A/I \approx \mathbb{C}$.

Proof. Suppose that A is a unital Banach algebra with $I \trianglelefteq A$ a maximal ideal. Then A/I is a simple unital Banach algebra (the proof of this actually comes later). Now let $b \in A/I$ be some non-invertible element. Then clearly $(A/I)b$ is an ideal, and since $1_{A/I} \notin (A/I)b$, it is a proper ideal. This means that $(A/I)b$ is the trivial ideal, which means that $b = 0_{A/I}$. Thus all non-zero elements of A/I are invertible. We have seen that this means that $A/I \approx \mathbb{C}$ □

2.2 Quotients

Proposition 2.9
If A is a Banach space (or algebra) and $I \trianglelefteq A$ is a closed ideal, then we can equip A/I with the *quotient norm*

$$||a + I|| = \inf_{z \in I} ||a + z||.$$

This norm makes A/I into a Banach space (or algebra).

Proof. FINISH ME □

We can note that $\pi : A \to A/I$ is a contraction as $||\pi(a)|| \leq ||a||$.

Proposition 2.10 (Factorization Lemma)
Let $\phi : A \to B$ be a homomorphism of Banach algebras. Then there is a factorization of $\phi = \dot{\phi} \circ \pi$, where $\pi : A \to A/\mathrm{Ker}(\phi)$ is the ususal quotient map and $\dot{\phi} : A/\mathrm{Ker}(\phi) \to B$. Further, we have that $||\dot{\phi}|| = ||\phi||$.

Proof. First we need to show that there exists some $\dot{\phi}$ that satisfies the proposition, then we will show that it is unique. Finally, we will show that $||\dot{\phi}|| = ||\phi||$.
Let $\dot{\phi}_1(a + \mathrm{Ker}(\phi) = \phi(a)$. To show that this is well defined note that if $a + I = b + I$, then $a = b + x$ for some $x \in \mathrm{Ker}(\phi)$. Thus $\dot{\phi}_1(a + I) = \phi(a) = \phi(a) + \phi(x) = \phi(a + x) = \phi(b) = \dot{\phi}_1(b + \mathrm{Ker}(\phi))$. Clearly $\dot{\phi}_1 \circ \pi = \phi$.
Now suppose that there is another function $\dot{\phi} : A/I \to B$ such that $\dot{\phi}_2 \circ \pi = \phi$. Then since π is surjective, $\dot{\phi}_1$ and $\dot{\phi}_2$ are equal on all values of A/I, meaning that they are equal.
Further, $||\phi|| = ||\dot{\phi} \circ \pi|| \leq ||\dot{\phi}|| \cdot ||\pi|| \leq ||\dot{\phi}||$ since π is a contraction. On the other hand $||\dot{\phi}(x + \mathrm{Ker}(\phi))|| = ||\phi(x)|| = ||\phi(x + z)|| \leq ||\phi|| \cdot ||x + z||$ for all $z \in \mathrm{Ker}(\phi)$. Thus $||\dot{\phi}(x + I)|| \leq ||\phi|| \inf_{z \in \mathrm{Ker}(\phi)} ||x + z|| = ||\phi|| \cdot ||x + \mathrm{Ker}(\phi)||$. Now use $x + I$ (maybe technically some $(x_n + I)$ approaching $x + I$) such that $||\dot{\phi}(x + I)|| = ||\dot{\phi}|| \cdot ||x + I||$. This shows that $||\dot{\phi}|| \leq ||\phi||$. Thus $||\dot{\phi}|| = ||\phi||$. □

3 Characters and the Gelfand Transformation

Definition 3.1
A *character* on a Banach algebra A is a non-zero homomorphism $\omega : A \to \mathbb{C}$.

Definition 3.2
The set of all characters on a Banach algebra A is called *the Gelfand Spectrum*, denoted $sp(A)$.

Lemma 3.3
Every character ω on a Banach algebra A is continuous. In fact, $||\omega|| \leq 1$, and if A is unital $||\omega|| = 1$, with $\omega(1_A) = 1$.

Proof. In order to show that ω is continuous we need to show that its norm is bounded. The in fact part of the lemma implies this so we will just prove that part.
Suppose for contradiction that for some character ω on a Banach algebra A, $||\omega|| > 1$ (possibly infinite). Then choose some $x \in A$ such that $||x|| \leq 1$ and $\omega(x) = 1$. We can find such an x, because since the norm of ω is greater than 1, we can find some x' such that $||x'|| < 1$ and $||\omega(x')|| > 1$. We can scale x' by the appropriate amount to get x. Then we define $y = \sum_{n=1}^{\infty} x^n$ (this series converges because the series of norms converges by the assumption that $||x|| < 1$). Then $y = x + xy$. So $\omega(y) = \omega(x) + \omega(x)\omega(y) = 1 + \omega(y)$. This gives the contradiction that $0 = 1$.
Now suppose that A is unital. Then find and $x \in A$ such that $\omega(x) \neq 0$. Then we get that $\omega(x) = \omega(1_A x) = \omega(1_A)\omega(x)$, so $\omega(1_A) = 1$. This gives that $||\omega|| \geq 1$, which means that they are equal by the previous part of the lemma. □

Example 3.4 (L)
t $A = C(\Omega)$ for some compact Ω. For any $t \in \Omega$, define $\omega_{:} C(\Omega) \to \mathbb{C}$ by $\omega_t(f) = f(t)$. Actually, it turns out that all characters on $C(\Omega)$ are of this form (proof to come later in course).

Here if Gelfand's idea:
Assumme that A is a unital Banach algebra, and the note that

$$\begin{aligned} sp(A) &= \{\omega : A \to \mathbb{C} \mid \omega \text{ is a non-zero homomorphism}\} \\ &\subseteq \{f : A \to \mathbb{C} \mid f \text{ is a linear functional and } ||f|| \leq 1\} \\ &= \text{the closed unit ball in the dual of } A,\ A^* \end{aligned}$$

A^* carries the weak *-topology, which is the topology generated by $U_{a_1,\cdots,a_n,\epsilon_1,\cdots\epsilon_n}(f) = \{f : A \to \mathbb{C} \mid ||g|| \leq \infty \text{ and } |g(a_i) - f(a_i)| < \epsilon_i\}$

This topology corresponds to a topology of point wise convergence, where $((f_n) \to f_n$ if and only if $(f_n(x)) \to f(x)$ for all $x \in A$. With this in mind, we define the Gelfand Transform.

Definition 3.5 (The Gelfand Transform)
The Gelfand Transform of a Banach algebra A is the function $\widehat{\cdot} : A \to C(sp(A))$ given by $x \mapsto \widehat{x}$ such that $\widehat{x}(\omega) = \omega(x)$

Theorem 3.6: The Gelfand Representation Theorem
Let A be a unital commutative Banach algebra. Then the following are true:

1. The Gelfand transformation is a continuous homomorphic function, and $||\widehat{x}||_\infty = r(x) \leq ||x||$.
2. The image of $\widehat{\cdot}$, $\widehat{A}$, separates points in $sp(A)$, meaning that if $\omega_1, \omega_2 \in sp(A)$ are not equal, then there is some $x \in A$ such that $\widehat{x}(\omega_1) \neq \widehat{x}(\omega_2)$.
3. For any $x \in A$, $\sigma_A(x) = \sigma_{C(sp(A))}(\widehat{x}) = \text{Im}(\widehat{x})$.

Proof.

1. Clearly the Gelfand transform is homomorphic, so all we need to show is that it is a bounded operator. That $||\widehat{x}||_\infty = r(x)$ follows easily from part 3, and since $r(x) \leq ||x||$ by theorem 1.15, we can conclude tat $||\widehat{\cdot}|| \leq 1$.
2. Obvious
3. It suffices to show that $x \in A$ is invertible if and only if $\widehat{x}$ is invertible. First suppose that $x \in A$ is invertible. Then for any character $\omega \in sp(A)$, $\omega(xx^{-1}) = \omega(x)\omega(x)^{-1} = 1$. Thus $\omega(x) \neq 0$ (since 0 is not invertible). Thus $\widehat{x}$ is no where 0, and this is equivalent to it being invertible (since we can construct its inverse).
   Now suppose that $x \in A$ is not invertible. We want to find some character ω such that $\widehat{x}(\omega) = \omega(x) = 0$, since this corresponds to $\widehat{x}$ not being invertible. Since x is not invertible, Ax is a proper ideal in A. We know that there is a maximal ideal $M \trianglelefteq A$ such that $Ax \trianglelefteq M$. In proposition 2.8 we have shown that $A/M \cong \mathbb{C}$. Thus we can treat the quotient map $\pi : A \to A/M$ as a character. Since $x \in M$ $\widehat{x}(\pi) = \pi(x) = 0$, meaning that $\widehat{x}$ is not invertible.

□

Definition 3.7
A Banach algebra A is *semisimple* if the Gelfand transform is injective

It turns out that A is semisimple if and only if the radical of A, $\text{rad}(A) = \{a \in A \mid r(x) = 0\}$ is trivial (only contains 0). Otherwise $A/\text{rad}(A)$ is semisimple.

It also turns out that all C^*-algebras are semisimple.

4 Commutative C^*-algebras and Functional Calculus

Even though the definition of an invoution does not require $||x|| = ||x^*||$ this will always be true in a C^*-algebra.

Definition 4.1

For a unital Banach algebra A, we define

$$e^x := \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

In order for this to be a valid definition we need that this series will always converge. Since the series of the norms converges and A is complete with respect to the norm, the series will converge.

In exercises, we have shwon that if $xy = yx$ then $e^{x+y} = e^x e^y$.

Definition 4.2
If A and B are involutive algebras, then a **-homomorphism* is a homomorphism $\phi : A \to B$ such that $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

The following theorem will be used to prove the 1st Gelfand-Naimark Theorem:

Theorem 4.3: Stone-Weirstrauss Theorem
If $C \subseteq C(\Omega)$ is a *-subalgebra for a compact Ω such that $1_{C(\Omega)} \in C$ and C seperates points in Ω, then the closure $\overline{C} = C(\Omega)$.

We do not prove this theorem.

Theorem 4.4: 1st Gelfand–Naimark Theorem
If A is a commutative unital C^*-algebra, then the Gelfand transform is a isometric *-isomorphism between A and $C(sp(A))$ (with norm $||\cdot||_\infty$ and involution given by $f^* = \overline{f}$)..

In this theorem, multiplication in $C(sp(A))$ is pointwise, the norm

Proof. The proof will proceed in three steps. First we will show that the Gelfand transform is a *-homomorphism. Next, we will show that it is an isomorphism, and finally we will show that it is an isometry.

1. We already know that the Gelfand transform is a homomorphism, so we only need to show that for any $x \in A$, $\widehat{x^*} = \widehat{x}^* = \overline{\widehat{x}}$. To do this we will first show that if $x \in A$ is self adjoint then for any $\omega \in sp(A)$, $\omega(x) \in \mathbb{R}$, which implies that $\widehat{x^*} = \overline{\widehat{x}}$. In an exercise we showed that any general element in A can be written as $x_1 + ix_2$ where x_1 and x_2 are self adjoint, which can show that $\widehat{\cdot}$ is a *-homomorphism.
   Let $x \in A$ be self adjoint. For any $t \in \mathbb{R}$, define $u_t = e^{itx}$. Then $u_t^{-1} = e^{-itx} = u_{-t}$. Using the definition of exponentiation, we can note that

   $$u_t^* = \sum_{n=0}^{\infty} \frac{1}{n!} \overline{(it)} x^* = \sum_{n=0}^{\infty} \frac{1}{n!} (-it)x = e^{-itx} = u_{-t}.$$

   Then $u_t u_t^* = 1_A$, and $||u_t||^2 = ||u_t^* u_t|| = 1$, so $||u_t|| = ||u_t^*|| = 1$. We also know that $\omega(u_t^* u_t) = 1$, so $|\omega(u_t^*)| \cdot |\omega(u_t)| = 1$. Since ω is a contraction (by lemma 3.3) this means that $|\omega(u_t)| = |\omega(u_t^*)| = 1$. We can write

$\omega(u_t) = \omega(\sum n = 0^\infty \frac{(it)^n)}{n!} x^n_= \sum n = 0^\infty \frac{(it)^n)}{n!} \omega(x)^n = e^{it\omega(x)}$. Then since $\omega(u_t) = e^i t \mathrm{Re}(\omega(x)) e^{-t\mathrm{Im}(\omega(x))} = 1$ for all t, it must be the case that $\mathrm{Im}(\omega(x)) = 0$ (if it were not then as $t \to \infty$, $\omega(u_t)$ grows unboundedly). Thus $\omega(x) \in \mathbb{R}$.

2. The image of the Gelfand transform $\widehat{A}$ is a *-subalgebra of $C(sp(A))$ such that $1_{C(sp(A))} \in \widehat{A}$ and $\widehat{A}$ seperates points in $C(sp(A))$ by theorem 3.6. Thus by the Stone-Weirstrauss Theorem, $\overline{\widehat{A}} = C(sp(A))$. On the other hand, we know that A is closed with respect to the norm and the Gelfand transform is an isometry (proved in the next part), $\widehat{A}$ must be closed, meaning that $\widehat{A} = \overline{\widehat{A}} = C(sp(A))$. Thus the Gelfand transform of A is surjective.
   To show injectivity, let $x, y \in A$ be distinct elements, then $||x - y|| > 0$. In the next part we show that the Gelfand transform of A is an isometry, which means that $|\omega(x - y)| = |\omega(x) - \omega(y)| > 0$. which implies that $\omega(x) \neq \omega(y)$.
3. In exercise we have shown that if $x^*x = xx^*$ (x is normal) then $||\widehat{x}||_\infty = r(x) = ||x||$. Since A is commutative, all elements are normal, so $\widehat{\cdot}$ is isometric *-isomorphism.

□

Corollary 4.5
If A is a unital C^*-algebra and $x \in A$ is self adjoint, then $\sigma(x) \in \mathbb{R}$.

Proof. Define $B = \overline{\{a_1 1_A + a_2 x + \cdots + a_n x^n \mod n \in \mathbb{N}, a_i \in \mathbb{C}\}}$. Then B is a unital commutative C^*-algebra. Then by the 1st Gelfand Naimark Theorem, $\widehat{\cdot} : B \to C(sp(B))$ is an isometric *-isomorphism. Following the first part of the proof of the theorem, we can see that $\widehat{x}$ is real valued. By The Gelfand Representation Theorem, we have that $\sigma_B(x) = \widehat{x}(sp(B) \subseteq \mathbb{R}$. We know that $\sigma_A(x) \subseteq \sigma_B(X) \subseteq \mathbb{R}$ by proposition 1.13. This proposition also tells us that $\delta\sigma_B(x) \subseteq \delta\sigma_A(x)$. But since $\sigma_B(x) \subseteq \mathbb{R}$ all of its elemebts are boundary points and thus $\sigma_A(x) = \sigma_B(X) \subseteq \mathbb{R}$. □

Lemma 4.6
In the setting from the above proof $\widehat{x} : sp(B) \to \sigma(x)$ is a homeomorphism.

Proof. We start by showing that $\widehat{x}$ is a bijection. We already know that it is surjective. To see injectivity, let $\omega, \chi \in sp(B)$ such that $\widehat{x}(\omega) = \widehat{x}(\chi)$. Then for any polynomial p, $\omega(p(x)) = \chi(p(x))$. Referring back to the definition of B, we can see that $\omega = \chi$ on a dense subset of B, and by continuity, this implies that $\omega = \chi$. □

We can form the C^*-algebra generated by 1 and x, $B := C^*(1, x)$ (B will be of the form that is presented in the proof of the lemma). This is a unital commutative C^*-algebra so we can apply the theorem to get that $x \mapsto \widehat{x}$ is an isomorphism showing that $B = C^*(1, x) \cong C(sp(B))$. Using the lemma we get that $sp(B) \cong \sigma(x)$, so we now know that $C^*(1, x) \cong C(C^*(1, x))$.

So we now have the following *-isomorphism between $C(\sigma(x))$ and $C(sp(C^*(1, x))$: $f \mapsto f \circ \widehat{x}$. As an exercise, it can be shown that the identity function maps to $\widehat{x}$ and that the uniformly 1 function maps to the uniformly 1 function. Thus we get the following theorem:

Theorem 4.7: Continuous Functional Calculus for Self Adjoint Elements
et A bd a unital C^*-algebra, and let $x \in A$ be a normal element. Then there exists an isometric *-isomorphism $\Phi_x : C(\sigma(x)) \to A$. wuth the following properties:

1. $\Phi_x(\mathrm{Id}_{C(\sigma(x))} = x$, and
2. $\Phi_x(1_{C(\sigma(x))} = 1_A$.

If $f \in C(\sigma(x))$, then $\Phi_x(f)$ can be denoted by $f(x)$. This can be thought of as a function of x inside of A.

The intuitive idea of functional calculus: If we are given a polynomial or any function that can be expressed as a power series it is quite clear how to apply that function to any algebra. The goal of functional calculus is to formally extend this idea to any continuous function.

Note in particular that for any normal x, if p is a polynomial, then $\Phi_x(p) = p(x)$ in the intuitive way. It can also be shown using the power series expansion that if $f : \mathbb{R} \to RR$ by $y \mapsto e^y$, then $f(x) = \Phi_x(f) = e^x$ with the definition of the exponential given above.

5 Representations of C^*-algebras

Definition 5.1
Let A be a C^*-algebra. A *representation* of A over a Hilbert space H is a *-homomorphism $\pi : A \to B(H)$. $\mathrm{rep}(A, H) = \{\pi : A \to B(H) \mid \pi \text{ is a representation}\}$.

Example 5.2
If $A = C[0, 1]$ and $H = L^2[0, 1]$, where $\langle \xi, \eta \rangle = \int_0^1 \xi(t)\eta(t)dt$. Then we can define $\pi : A \to H$ by $(\pi(f)\xi)(t) = f(t)\xi(t)$.

In general we only consider non-degenerate representations, i.e. π such that $\pi(A)H = \overline{span}\{\pi(a)\xi \mid a \in \underline{A, \xi \in H}\} = H$. If this fails then we can simply restrict the representation to the subspace $\overline{\pi(A)H}$ to make it non-degenerate (restricting means only applying $\pi(a)$ to elements of the subspace, so the restriction is from A to the set of bounded linear operators on the subspace).

Definition 5.3
Given a C^*-algebra A, Hilbert space H, and a representation $\pi : A \to B(H)$, $K \subseteq H$ is an *invariant subspace* of π if $\pi(a)K \subseteq K$ for all $a \in A$.

If K is an invariant subspace of π then the restriction $\pi|_K : A \to B(K)$ where $\pi|_K(a) = \pi(a)|_K$ is a representation of A over K.

Example 5.4
If we fix some $\xi \in H$ then the space $K = \overline{\pi(A)\xi} \subseteq H$ is an invariant subspace.

Definition 5.5
Given finitely many representations $\pi_i \in \text{rep}(A, H_i)$ for $1 \leq i \leq n$, we can form the *direct sum of representations* $\oplus_{1 \leq i \leq n} \pi_i : A \to \oplus_{1 \leq i \leq n} H_i$, where for any $a \in A$, $a \mapsto \oplus_{1 \leq i \leq n} \pi_i(a)$, such that $\oplus_{1 \leq i \leq n} \pi_i(a)(\xi_1, ..., \xi_n) = (\pi_i(\xi_i))_{1 \leq i \leq n}$.

In the infinite case, more care must be taken. Let I be an infinite index set. Then define $\oplus_{i \in I} H_i = \{(\xi_i)_{i \in I} \mid \xi_i \neq 0 \text{ for finitely many } i, \text{ and } _{i \in I} ||\xi_i||^2 \leq \infty\}$. Then $\oplus_{i \in I} \pi_i : A \to \oplus_{i \in I} H_i$, where for any $a \in A$, $a \mapsto \oplus_{i \in I} \pi_i(a)$, such that $\oplus_{i \in I} \pi_i(a)(\xi_i) = (\pi_i(\xi_i))_{i \in I}$ for $(\xi_i) \in \oplus_{i \in I} H_i$.

Although it is not relevant here, it is interesting to know some following things about representations. A representation if said to be *irreducible* if its only invariant subspaces are the full space and the empty set. For any representation, $H = \text{Ker}(\pi) \oplus \text{Im}(\pi)$. Any representation can be written as the direct sum of irreducible representations .

Definition 5.6
A representation $\pi \in \text{rep}(A, H)$ is said to be *faithful* if it is inejctive.

Lemma 5.7
Suppose that A is a unital C^*-algebra, and for some fixed $x \in A$, $g \in C(\sigma(x))$. Now suppose that π is some *-homomorphism with domain A such that $||\pi|| < 1$. Then $\pi(g(x)) = g(\pi(x))$.

Proof. The lemma is clearly true in the case that g is some polynomial (because of the homomorphism of π). By the Wierstrauss approximation theorem, we know that polynomials are dense in $C(\sigma(x))$. Since $||\pi|| < 1$, this means that we can extend the result from polynomials to general continuous functions. □

Theorem 5.8
If A is a unital C^*-algebra and $\pi \in \text{rep}(A, H)$, then $||\pi|| \leq 1$. Moreover, if π is faithful, then $||\pi|| = 1$ and π is an isometry.

Note that this theorem is true for general *-homomorphisms, not just representations.

Proof. Assume that π is non-degenerate. (If it is then simply restrict to the appropriate subspace to make it non-degenerate. We claim that $\sigma_{B(H)}(\pi(a)) \subseteq \sigma_A(a)$. This is equivalent to $\mathbb{C} \setminus \sigma_A(a) \subseteq \mathbb{C} \setminus \sigma_{B(H)}(\pi(a))$. Then suppose $\lambda \in C \setminus \sigma_A(a)$. Then $(\lambda 1_A - a)^{-1} \in A$ exists, and $\pi((\lambda 1_A - a)^{-1}) = (\lambda \pi(1_A) - \pi(a))^{-1} = (\lambda 1_{B(H)} - a)^{-1}$, so $\lambda 1_{B(H)} - a$ is invertible and $\lambda \in \sigma_{B(H)}(\pi(a))$. This proves the claim.
Now, for any $a \in A$ the element $a^* a$ is self adjoint, and we have that

$$\begin{aligned} ||a||^2 = ||a^* a|| = r(a^* a) &= \sup\{|\lambda| \mid \lambda \in \sigma_A(a^* a)\} \\ &\leq \sup\{|\lambda| \mid \lambda \in \sigma_{B(H)}(\pi(a))\} \\ &= r(\pi(a^* a)) = ||\pi(a)^* \pi(a)|| = ||\pi(a)||^2. \end{aligned}$$

Thus $||a|| \leq ||\pi(a)||$, and $||\pi|| \leq 1$.

For the moreover part of the theorem, suppose that π is injective. We want to show that $||\pi(a)|| = ||a||$. For any $a \in A$, let $x = a^*a$, and let $X = \pi(x)$. Claim that $||x|| = ||X||$. If this is true then $||a^||2 = ||x|| = ||X|| = ||pi(a)||^2$, which proves the theorem. Suppose for contradiction that $||x|| > ||X||$ for some $a \in A$ where $x = a^*a$. Then we have that $r(x) = ||x|| < ||X|| = r(X)$. Thus we have that $\sigma_{B(H)}(X) \subset \sigma_A(x)$ with proper containment. Since x and X are self adjoint, their spectrums are subsets of the reals. Then it is true that we can find some continuous positive function $f : \sigma_A(x) \to \mathbb{R}_{\geq 0}$ such that $f|_{sigma_{B(H)}(X)} = 0$ but f in general is not uniformly 0. Using the previous lemma, we know that $\pi(f(x)) = f(\pi(x))$. Then $f(\pi(x)) = f(X) = 0_{B(H)}$. On the other hand, since f is non zero, $f(x) \neq 0$, which means that $\pi(f(x)) \neq 0_{B(H)}$ by injectivity. This contradicts the result of the lemma. Thus we conclude that π must be an isometry. □

6 The GNS Construction

GNS-Gelfand Nairmark Segal

We will build up to a general method for constructing *-representations. The results and construction below hold for general Banach *-algebras, so it is true specifically for C^*-algebras.

Definition 6.1 (L)

t A be a Banach *-algebra.

1. $y \in A$ us *positive* if $y = x^*x$ for some $x \in A$.
2. A linear functional $\rho : A \to \mathbb{C}$ is *positive* if $\rho(x^*x) \geq 0$ for any $x \in A$.
3. If A is a unital Banach *-algebra and ρ is a positive linear functional such that $\rho(1_A) = 1$, then we say that ρ is a *state*.

Example 6.2

1. If A is a commutative C^*-algebra, then we can write $A = C(\Omega)$ for some compact Ω. $y \in A$ is positive means that $y = x^*x = \overline{x}x$ for some x. Thus y is positive if and only if $y \geq 0$.
2. Let $\rho : C[0,1] \to \mathbb{C}$ be defined by $\rho(f) = \int_0^1 f(t)r(t)dt$ is positive if and only if $r(t) > 0$. ρ is a state if $\int_0^1 r(t)dt = 1$.

A_+ denotes the positive elements of A. If $x \in A_+$ then we write $x \geq 0$. We can also say that $x \geq y$ if and only if $x - y \geq 0$. (Note that $x \not\geq 0$ does not mean that $x < 0$ in this notation I THINK). This gives us a partial ordering on A. As we showed in the example part 2, in $C(\Omega)$, we have that $x \geq 0$ if and only if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$. It will turn out that this is true in general (proof later). Thus if $x \in A_+$ we can define x using functional calculus. $S(A)$ is the set of states on A.

Lemma 6.3

If A is a Banach *-algebra and $x \in A$ is self adjoint and $||x|| \leq 1$, then there is a $y \in A$ such that $y^2 = 1 - x$ and y is self adjoint.

Proof. The proof uses taylor series expansions and holomorphic functional calculus □

Proposition 6.4 (Cauchy-Schwartz Inequality for Positive Linear Functionals)
Let A be a Banach *-algebra, and let $\rho : A \to \mathbb{C}$ be a positive linear functional. Define the sequilinear form $[x, y] = \rho(y^*x)$ Then $|[x, y]|^2 \leq [x, x][y, y]$. More over, $||\rho|| = \rho(1)$.

Proof. To prove the first part of the proposition note that the sesquilinear form is almost an inner product, except that we dont have $[x, x] = 0 \iff x = 0_A$. However, you can still follow your favorite proof of the Cauchy-Schwartz inequality to get the result.
Now we prove that $||\rho|| = \rho(1)$. Since ρ is positive, we have that $\rho(1_A) = \rho(1_A^* 1_A) \geq 0$. The first part of the proof shows that $|\rho(x)|^2 = |\rho(1_A^* x)|^2 \leq \rho(1)\rho(x^*x)$. Now we claim that if$||x|| < 1$ then $\rho(x^*x) \leq \rho(1)$. If this is the case then $|\rho(x)| \leq \rho(1_A)$ and $|||| \leq \rho(1_A)$. On the other hand we obviously have that $\rho(1_A) \leq ||\rho||$. To prove the claim first not athat $||x^*x|| = ||x||^2 < 1$. Then x^*x is self adjoint and we can apply the lemma to find some $y = y^* \in A$ such that $y^2 = 1 - x^*x$. Then $0 \leq \phi(y^2) = \rho(y^*y) = \rho(1_{A-}\rho(x^*x)$. This $\rho(x^*x) \leq \rho(1_A)$, which proves the claim and thus the proposition. □

Lemma 6.5
Let A be a Banach *-algebra, let $\rho : A \to \mathbb{C}$ be a positive linear functional, and let$N = \{x \in A \mid [x, x] = \rho(x^*x) = 0\}$. Then N is a closed left-ideal in A, and the the function $\langle x + N, y + n\rangle = [x, y]$ is a well defined inner product on A/N.

Proof. First we show that N is closed. Let (x_n) be a sequence in N that has limit x. Then $(x_n^*x) \to x^*x$ and thus $(0) = \rho(x_n^*x) \to \rho(x^*x)$, so $0 = \rho(x^*x)$ and $x \in N$.
Now we show that N is a left ideal. Let $x \in N$ and $a \in A$. Then we consider $\rho((ax)^*ax) = \rho(x^*a^*ax)$. By the Cauchy-Schwartz inequality, we have that $|\rho(x^*a^*ax)|^2 \leq \rho(a^*a)\rho((a^*ax)^*(a^*ax))$. Since $x \in N$ the right hand side is 0. Since ρ is positive, this means that $\rho((ax)^*ax) = 0$, so $ax \in N$.
The second assertion breaks into 4 parts:

1. Well defined: Let $x_1 + N = x_2 + N \in A/N$ and $y_1 + N = y_2 + N \in A/N$. We want to show that $\langle x_1 + N, y + N\rangle = \langle x_2 + N, y_2 + N\rangle$. We know that for some $x, y \in N$, $x_2 = x_1 + x$ and $y_2 = y_1 + y$. Then we have that $\langle x_2 + N, y_2 + N\rangle = [x_2, y_2] = [x_1 + x, y_1 + y]$ if we expand this and apply the Cauchy-Schwartz inequality to each of the terms that we want to cancel, we will find that this simplifies to $[x_1, y_1] = \langle x_1 + N, y_1 + N\rangle$.
2. Linear in the first argument: follows easily from the linearity of ρ.
3. Conjugate symmetric: $\rho((x + y)^*(x + y)) \in \mathbb{R}$. When we expand this, it implies that $\rho(x^*y) + \rho(y^*x) \in \mathbb{R}$. The only way this can be true for all x, y is if they are conjugate to each other. Thus the proposed inner product is conjugate symmetric.
4. $\langle x + N, x + N\rangle = [x, x] \geq 0$ since ρ is positive. Furthermore, $\langle x + N, x + N\rangle = 0$ implies that $x \in N$ which means that $x + N = 0_{A/N}$.

□