An Introduction to the Theory of Infinity Categories

by

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Abstract This project seeks to understand ∞-categories (also known as quasi-categories) as categories in which the notion of equality of both objects and morphisms has been replaced by isomorphism. To do this, we think of ∞-categories as mathematical structures with objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, and so on. Thus we can consider whether there is an isomorphism between two n-morphisms, rather than considering if they are equal. We show that there is a fully faithful functor from the category of small categories to the category of ∞-categories, which can be used to generalize many important categorical constructions to the setting of ∞-categories. In addition, we show that the study of a particular type of ∞-category, called a Kan complex, is equivalent to the study of the homotopy theory of spaces.

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Chapter 0

Introduction

When introduced to category theory, students are often told to not consider the equality of the objects, as the structure of the objects of a category does not contribute to the structure of the category itself. This idea breaks down when we consider Cat, the category of small categories and functors between them. In order to discern if two morphisms $F, G \colon \mathcal{C} \to \mathcal{D}$ in Cat are equal, one must check if they are equal on the objects of $\mathcal{C}$, which requires considering the equality of objects in $\mathcal{D}$. However, we just asserted that the equality of objects is unimportant to understanding the structure of $\mathcal{D}$ as a category. This is perplexing as functors are morphisms between categories that preserve structure, but determining if two functors are equal requires more than just structure of the categories in question.

One solution would be to just ignore the question of whether or not two functors are equal. We already have the notion of natural transformations and natural isomorphisms of functors, and instead of asking if two functors are equal, we could decide to only ever ask if there is a natural isomorphism from one functor to another. This idea generalizes to the concept of a (strict) 2-category, which essentially encodes the idea of having objects, morphisms between objects, and morphisms between morphisms which are called 2-morphisms. (More formally a strict 2-category is a category enriched over Cat.) For example, Cat can be turned into a 2-category where the objects are small categories, the morphisms are functors, and the 2-morphisms are natural transformations of functors.

Ignoring the notion of equality of objects has forced us to ignore the equality of morphisms, and instead only consider isomorphisms between morphisms. However, answering the question of whether two natural transformations are equal requires considering the equality of objects of a category. So one could go one level deeper and use the notion of a strict 3-category, and then of a strict 4-category, and continue indefinitely. ∞-categories are a structure that allow one to consider objects, morphisms of morphisms, morphisms of morphisms of morphisms, and so on forever. This frees us from ever having to answer the question of whether two morphisms of degree n are equal, as we can instead ask if there is an $n+1$ degree isomorphism between them.

Getting rid of the notion of equality introduces many difficulties, but also brings a certain cleanliness. For example, in the category of groups we have groups that are isomorphic but not equal. We can get around this issue by constructing the ∞-category of groups in which we refuse to consider the question of equality, and only ask if two groups (or group morphisms, morphisms of morphisms, etc.) are isomorphic.

The model for ∞-categories discussed in this project is based on simplicial sets and requires a substantial amount of background, which is the subject of Chapter 1. In Chapter 2, we introduce the nerve functor, which assigns to any category a simplicial set, and the singular functor which assigns to any topological space a simplicial set. The nerve of a category and the singular set of a space are both examples of ∞-categories. Since there is already a notion of isomorphism in topological homotopy theory, namely homotopy, the singular functor will motivate our definition of isomorphism in an ∞-category, which we will also call homotopy. Ordinary category theory is undeniably useful, so we want to generalize constructions from ordinary category theory to the setting of ∞-categories, where we can not use equality. The nerve functor will be useful in motivating many of these generalizations. Chapter 3 defines ∞-categories, introduces the concept of homotopy and also generalizes many of the notions of ordinary category theory to the context of ∞-categories. Finally, in Chapter 4, we discuss various types of fibrations and their application to the study of ∞-categories.

Chapter 1

Background

1.0 Notation

Set	Category of sets and functions between sets
Top	Category of compactly generated Hausdorff spaces and continuous maps
Cat	Category of small categories and functors between small categories
$\mathcal{D}^{\mathcal{C}}$	Category of functors from $\mathcal{C}$ to $\mathcal{D}$

In this paper all of the topological spaces are objects of Top. Similarly, a generic category is always assumed to be an object of Cat

1.1 The Yoneda Lemma and the Density Theorem

Let $\mathcal{C}$ be any small category. For any object A, the functor $\mathrm{Hom}_{\mathcal{C}}(-, A)\colon \mathcal{C}^{op} \to \mathsf{Set}$ is the functor represented by A, which we denote by h_A. Given another functor $F\colon \mathcal{C}^{op} \to \mathsf{Set}$, the set of natural transformations from h_A to F is $\mathrm{Hom}(h_A, F)$.

Lemma 1.1.0.1 (Yoneda Lemma)**.** *For any small category $\mathcal{C}$, object A of $\mathcal{C}$, and functor $F\colon \mathcal{C}^{op} \to \mathsf{Set}$ there exists an isomorphism of sets $Hom(h_A, F) \cong F(A)$. Moreover, when both sides are viewed as functors $\mathcal{C}^{op} \times \mathsf{Set}^{\mathcal{C}^{op}} \to \mathsf{Set}$, the isomorphism is natural in both A and F.*

Proof. Define $p\colon \mathcal{C}^{op} \times \mathsf{Set}^{\mathcal{C}^{op}} \to \mathsf{Set}$ as follows.

- For any object (h_A, F), let $p(A, F) = \mathrm{Hom}(h_A, F)$.
- For any morphism $(u^{op}, v)\colon (A, F) \to (B, G)$, let $p(u^{op}, v)(\eta) = v \circ \eta \circ u$, where $\eta \in \mathrm{Hom}(h_A, F)$.

Define $q\colon \mathcal{C}^{op} \times \mathsf{Set}^{\mathcal{C}^{op}} \to \mathsf{Set}$ as follows.

- For any object (A, F), let $p(A, F) = F(A)$.
- For any morphism (u^{op}, v) as above, let $p(u^{op}, v)(x) = v_B(F(u^{op})(F(A)))$ where $x \in F(A)$.

For an arbitrary object (A, F) of $\mathcal{C}^{op} \times \mathsf{Set}^{\mathcal{C}^{op}}$, define $\Phi_{A,F}\colon p(A, F) \to q(A, F)$ by the mapping $\eta \mapsto \eta_A(id_A)$. We must show that the collection $\{\Phi_{A,F}\}_{(A,F) \in \mathrm{Ob}(\mathcal{C}^{op} \times \mathsf{Set}^{\mathcal{C}^{op}})}$ is a natural transformation from p to q and that for a given pair (A, F) the morphism $\Phi_{A,F}$ is a bijection.

We will start by showing the second part. We have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[rrr, "f\circ"] \arrow[ddd, "\eta_A"'] & & & \mathrm{Hom}(X,A) \arrow[ddd, "\eta_X"] \\
& id_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& \eta_A(id_A) \arrow[r, mapsto] & \eta_X(f) & \\
F(A) \arrow[rrr, "F(f)"] & & & F(X)
\end{tikzcd}$$

Any natural transformation $\eta \in p(A, F)$ is uniquely determined by $\eta_A(id_A)$, since $\eta_X(f) = F(f)(\eta_A(id_A))$ for any $f \in \mathrm{Hom}(X, A)$. Thus $\Phi_{A,F}$ is a bijection between natural transformations from h_A to F and elements of $F(A)$.

Now we show that Φ is a natural transformation from p to q. We must show that the following diagram commutes.

$$\begin{tikzcd}
p(A,F)=\mathrm{Hom}(h_A,F) \arrow[rrr, "p(u^{op},v)"] \arrow[ddd, "\Phi_{A,F}"] & & & p(B,G)=\mathrm{Hom}(h_B,G) \arrow[ddd, "\Phi_{B,G}"] \\
& \eta \arrow[r, mapsto] \arrow[d, mapsto] & v\circ\eta\circ u \arrow[d, mapsto] & \\
& \eta_A(id_A) \arrow[r, mapsto] & \begin{array}{c}(v\circ\eta\circ u)_B(id_B)\\ =v_B(F(u^{op})(\eta_A(id_A)))\end{array} & \\
q(A,F)=F(A) \arrow[rrr, "(q*u^{op},v)"] & & & q(B,G)=G(B)
\end{tikzcd}$$

Since $u^{op} = \circ u$ sends $id_A \in \mathrm{Hom}(A, A)$ to $u \in \mathrm{Hom}(B, A)$, by the previous argument, we have that $F(u^{op})(\Phi_{A,F}(\eta)) = F(u^{op})(\eta_A(id_A)) = \eta_B(u)$. This gives
$v_B(F(u^{op})(\eta_A(id_A))) = v_B(\eta_B(u)) = v_V(\eta_B(u(id_B)))$, as desired. $\square$

The Yoneda Embedding Let $y\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{op}}$ be the functor given by $y(A) = h_A$. The Yoneda Lemma shows that

$$\begin{aligned}
\mathrm{Hom}(y(A), y(B)) &= \mathrm{Hom}(h_A, y(B)) \\
&\cong y(B)(A) \\
&= h_B(A) \\
&= \mathrm{Hom}_{\mathcal{C}}(A, B).
\end{aligned}$$

Thus y is fully faithful and gives an embedding of $\mathcal{C}$ in $\mathsf{Set}^{\mathcal{C}^{op}}$, called the Yoneda embedding.

1.1.1 The Density Theorem

Let $\mathcal{C}$ be a small category. Informally, the density theorem says that any functor $\mathcal{C}^{op} \to \mathsf{Set}$ can be realized as a colimit of representable functors.

Theorem 1.1.1.1 (The Density Theorem)**.** *Let $F\colon \mathcal{C}^{op} \to \mathsf{Set}$ be any functor. Define $\mathcal{I}_F$, the category of elements of F, to be the category where*

- *objects are pairs (U, s) where U is an object of $\mathcal{C}$ and $x \in F(U)$, and*
- *morphisms $(U, x) \to (V, y)$ correspond to morphisms $u\colon U \to V$ such that $(F(u))(y) = x$.*

Let $p\colon \mathcal{I}_F \to \mathcal{C}$ be the forgetful functor. Then F is isomorphic to the colimit of the diagram $y \circ p\colon \mathcal{I}_F \to \mathsf{Set}^{\mathcal{C}^{op}}$.

For brevity we do not prove the density theorem, but a proof can be found in 6.2.17 of [Lei09].

Specifying an object (U, x) of $\mathcal{I}_F$ is essentially specifying $x \in F(U)$, which is equivalent to specifying the natural transformation $\Phi^{-1}_{U,F}(x)$ from h_U to F. By the naturality of Φ, specifying a morphism $(U, x) \to (V, y)$ is equivalent to specifying a map $\mathrm{Hom}(h_V, F) \to \mathrm{Hom}(h_U, F)$ that takes the natural transformation $\Phi^{-1}_{V,F}(V, y)$ to the natural transformation $\Phi^{-1}_{U,F}(U, x)$. Thus we can equivalently define $\mathcal{I}_F$ as having objects that are natural transformations from h_U to F for all objects U of $\mathcal{C}$, and morphisms from $\eta : h_U \to F$ to $\varphi : h_V \to F$ corresponding to maps $U \to V$ that take φ to η by precomposition. In this construction we have the forgetful functor p' that takes a natural transformation from h_U to F to U and the statement of the density theorem is that F is isomorphic to the colimit of $y \circ p'$.

1.2 Geometric Intuition

Before giving the definition of a simplicial set, we start by considering geometric simplicial complexes. The goal is to gain some intuition for the behavior of simplicial sets.

A geometric n-simplex is a convex set spanned by $n+1$ geometrically independent points, and any n-element subset is a face. A geometric simplicial complex X is a collection of geometric simplices of various dimension such that

- every face of every simplex of X is in X, and
- the intersection of any two simplices of X is a face of each simplex.

One can specify a geometric simplicial complex X by starting with a set of vertices and then specifying which sets of vertices span a simplex of X. we denote a simplex that has vertices $\{v_1, \ldots, v_n\}$, by $(v_1, \ldots, v_n)$. In this case, any subset of $\{v_1, \ldots, v_n\}$ must also span a simplex. One can organize X as skeleta $X^0, X^1, \ldots$ where X^0 is the collection of vertices and X^n is all of the n-element subsets of X^0 that span a simplex of X.

Example 1.2.0.1. We can give the standard topological n-simplex
$|\Delta^n| := \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0\}$ the structure of a geometric simplicial complex. Let $X^0 = \{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots (0, 0, \ldots, 1)\}$. Then for any $k \leq n$, let X^k by the sets $k+1$ element subsets of X^0.

Given two geometric simplicial complexes X and Y, if there is a map $X^0 \to Y^0$ that induces an isomorphism $X^k \to Y^k$ for all integers $k \geq 0$, then X and Y are homeomorphic when considered as spaces. The idea of an abstract simplicial set is to strip away the geometric data so we are left with a purely combinatorial structure that can describe any geometric simplicial complex up to homeomorphism.

An abstract simplicial complex X consists of a set X^0, called the vertices of X, and for each integer $k > 0$ X^k consists of $k+1$ element subsets of X^0 such that for any integer $j \leq k$ any $j+1$ element subset of an element of X^k is an element of X^j. We call X^k the set of abstract k-simplices of X.

We define a face of an abstract n-simplex as any n-element subset. One could create a geometric simplicial complex from an abstract simplicial complex X of finite dimension by giving an appropriate map $X^0 \to \mathbb{R}^N$ that would take abstract k-simplices to geometric k-simplices in such a way that the intersection of any two geometric k-simplices is a face of both of them.

Example 1.2.0.2. Any geometric simplicial complex is an abstract simplicial complex. To emphasize the lack of geometric information we construct a combinatorial generalization of the standard topological n-simplex where X^0 is any set of $n+1$ elements, and for $k \leq n$, X^k is the set of $k+1$ element subsets of X^0.

The lack of an ordering on the vertices make abstract simplicial complexes difficult to work with. For example, there is no natural ordering of the $n+1$ faces of an abstract n-simplex. To address this we move to ordered simplicial complexes.

An ordered simplicial complex is an abstract simplicial complex X along with a total ordering on X^0. Then we can specify k-simplices of X by $(k+1)$-tuples $(x_{j_0} < x_{j_1} < \cdots < x_{j_k})$. We define the ith face of such a k-simplex as $(x_{j_0} < \cdots < \hat{x_{j_i}} < \cdots < x_{j_n})$, the tuple with the ith entry removed. Ordered simplicial complexes give a combinatorial analouge to a geometric simplicial complex with the additional structure of an ordering on the vertices and faces of any particular k-simplex.

1.3 The Category Δ

Given an ordered abstract simplicial complex, one is able to give an ordering to the $n+1$ faces of any n-simplex. We want to encode in a categorical way the idea of a simplex having ordered faces. For this we will need the following category.

Definition 1.3.0.1. Let Δ to be the category where

- objects are sets $[n] = \{0, 1, \cdots, n\}$ for each integer $n \geq 0$, and
- morphisms are the weakly order preserving functions.

For any partially ordered set I, we can form a category that has objects corresponding to the elements of I and a morphism $i \to j$ when $i \leq j$. Weakly order preserving maps between partially ordered sets are equivalent to functors between the corresponding categories in the obvious way. Thus we can equivalently define Δ as the full subcategory of Cat that is spanned by the categories $[n]$ such that $n \geq 0$. In different contexts different definitions will be more convenient so we use them interchangeably.

Although there are many possible morphisms $[n] \to [m]$, for integers $n, m \geq 0$, it turns out that we can express any such morphism as the composition of two types of simple morphisms. For any $0 \leq i \leq n$, define $d^i \colon [n] \to [n+1]$ and $s^i \colon [n] \to [n-1]$ as:

$$d^i(k) = \begin{cases} k, & \text{if } k < i \\ k+1, & \text{if } k \geq i \end{cases} \qquad \text{and} \qquad s^i(k) = \begin{cases} k, & \text{if } k \leq i \\ k-1, & \text{if } k > i \end{cases}$$

These maps are called the *coface maps* and *codegeneracy maps*, respectively. The ith coface maps sends $[n]$ to $[n+1]$ in order by skipping i in $[n+1]$, while the ith codegeneracy map sends $[n]$ to $[n-1]$ by repeating i. There are a number of relations that are easy to verify:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \text{ for } i < j \\ s^j s^i &= s^i s^{j+1} \text{ for } i \leq j \\ s^j d^i &= \begin{cases} id, & \text{if } i = j \text{ or } \text{ i=j+1} \\ d^i s^{j-1}, & \text{if } i < j \\ d^{i-1} s^j, & \text{if } i > j+1 \end{cases} \end{aligned} \tag{1.1}$$

The coface and codegeneracy maps generate all of the morphisms of Δ, and using the relations above, one can express any morphism as a composition of coface and codegeneracy maps.

When considering Δ as a subcategory of Cat, the coface and codegeneracy maps are defined in the following way. We depict the category $[n]$ as $(0 \xrightarrow{f_1} 1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} n)$. Then

- $d^i \colon [n] \to [n+1]$ is defined by $d^i(f_k) = \begin{cases} f_k, & \text{if } k < i \\ f_{i+1} \circ f_i & \text{if } k = i \\ f_{k+1}, & \text{if } k > i \end{cases}$
- $s^i \colon [n] \to [n-1]$ is defined by $s^i(f_k) = \begin{cases} f_k, & \text{if } k < i \\ id_k & \text{if } k = i \\ f_{k-1}, & \text{if } k > i \end{cases}$.

As with the original definition of Δ, any morphism in the categorical version Δ can be uniquely expressed as a composition of these morphisms.

1.4 Simplicial Sets

Definition 1.4.0.1. A functor $\Delta^{op} \to \mathsf{Set}$ is called a simplicial set.

If X is a simplicial set, the we denote by X_n the image $X([n])$ and refer to elements of X_n as n-simplices of X. For any integers $0 \leq i \leq n$ the ith *face* and *degeneracy maps* of X, are defined as $d_i^X = X(d^i) \colon X_{n+1} \to X_n$ and $s_i^X = X(s^i) \colon X_{n-1} \to X_n$, respectively. We omit the super script when there is no ambiguity regarding which simplicial set is being considered. For any n-simplex x of X, we refer to $d_i(x)$ as the ith face of x and $s_i(x)$ as the ith degeneracy of x. Since the morphisms d^i and s^i generate all of the morphisms of Δ, specifying the face and degeneracy maps as well as the sets X_n is sufficient to uniquely identify a simplicial set.

The face and degeneracy maps morphisms satisfy the duals of the relations (1.1). Thus one can equivalently define a simplicial set X as a collection of sets X_n for each integer $n \geq 0$ along with morphisms $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_{n-1} \to X_n$ for each $0 \leq i \leq n$ that satisfy the following relations:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \text{ for } i < j \\ s_i s_j &= s_{j+1} s_i \text{ for} i \leq j \\ d_i s_j &= \begin{cases} id, & \text{if } i = j \text{ or } i = j+1 \\ s_{j-1} d_i, & \text{if } i < j \\ s_j d_{i-1}, & \text{if } i > j+1 \end{cases} \end{aligned} \tag{1.2}$$

Example 1.4.0.2. Let X be an abstract ordered simplicial complex. Then for any n-simplex $(v_{j_0}, \dots, v_{j_n})$, let $d_i(v_{j_0}, \dots, v_{j_n}) = (v_{j_0}, \dots, \hat{v_{j_i}}, \dots, v_{j_n})$, and let $s_i(v_{j_0}, \dots, v_{j_n}) = (v_{j_0}, \dots, v_{j_i}, v_{j_i}, \dots, v_{j_n})$. By adjoining $n+1$ element tuples that have repeated elements to X_n, we get simplicial set $\Delta^{op} \to \mathsf{Set}$ where $[n] \mapsto X^n$, $d^i \mapsto d_i$, and $s^i \mapsto s_i$. Note that the definition of the ith face of a simplex in an ordered simplicial complex aligns with the definition of the ith face when included into this simplicial set.

If we turn the simplicial set from this example into a geometric simplicial complex as discussed in Section 1.2, the ith degeneracy of any n-simplex (which is an $n+1$-simplex) would be sent to a geometric n-simplex. We don't like these garbage annoying simplices, so we call any simplex that is the image of some s_i a degenerate simplex.

Geometric simplices form the basic building blocks of geometric simplicial complexes, and we now introduce the corresponding notion for simplicial sets, the standard simplices. These simplicial sets will form the basic building block of simplicial sets. In 1.4.1 we will rigorously define how one can construct any simplicial set using the standard simplices.

Definition 1.4.0.3. For any integer $n \geq 0$, we call $\Delta^n := \mathrm{Hom}_\Delta(-, [n])$ the nth standard simplex.

For any integer $k \geq 0$, the set of k-simplices, Δ^n_k, is the collection of the morphisms $[k] \to [n]$ in Δ. The face map $d_i \colon \Delta^n_{k+1} \to \Delta^n_k$ sends a $(k+1)$-simplex f to $f \circ d^i$, and the degeneracy map $s_i \colon \Delta^n_{k-1} \to \Delta^n_k$ sends a $(k-1)$-simplex g to $g \circ s^i$. For $0 \leq k \leq n$, the non-degenerate k-simplices of Δ^n_k are the injective maps $[k] \to [n]$. For $k > n$ all k-simplices of Δ^n are degenerate (since none of these maps are injective).

We can identify any k-simplex of Δ^n with the $(k+1)$-tuple $(f(0), \dots f(k))$. This gives a graphical way to represent the nth standard simplex. Start the standard topological n-simplex that has $n+1$ vertices labeled from 0 through n. Then let there be an arrow from the ith to the jth vertex when $i \leq j$. Any k-simplex of Δ^n corresponds to the convex hull spanned the vertices $\sigma(0) \leq \dots \leq \sigma(k)$. Conversely, any ordered list of $k+1$ vertices $v_0 \leq \dots \leq v_k$, there is a corresponding k simplex of Δ^n that sends i to v_i. A simplex is degenerate if and only if there is a repeated vertex in the corresponding list. The ith face map, $d_i\Delta^n$ corresponds to projecting the topological n-simplex on to its face opposite the ith vertex.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Figure 1.1: A depiction of Δ^2. The tuple (0,0,2) represents the 2-simplex $(0, 1, 2) \mapsto (0, 0, 2)$.

Remark 1.4.0.4. Given the categorical definition of the category Δ, the corresponding definition of Δ^n is $\Delta^n = \mathrm{Hom}_{\mathsf{Cat}}(-, [n])$, where for any integer $k \geq 0$, the k-simplices are $\Delta^n_k = \mathrm{Hom}_{\mathsf{Cat}}([k], [n]) = N([n])_k$. Thus $\Delta^n = N([n])$ when using the categorical definition of Δ.

Definition 1.4.0.5 (Simplicial Maps)**.** The category of simplicial sets, denoted sSet, is the functor category $\mathsf{Set}^{\Delta^{op}}$. Given two simplicial sets $X, Y \colon \Delta^{op} \to \mathsf{Set}$, a simplicial set morphism from X to Y is a natural transformation from X to Y.

We can equivalently define a simplicial set morphism $f \colon X \to Y$ as a collection of maps $f_n \colon X_n \to Y_n$ such that $f_{n-1} \circ d_i^X = d_i^Y \circ f_n$ and $f_n \circ s_i^X = s_i^Y \circ f_{n-1}$.

Example 1.4.0.6. For any morphism $g \colon [n] \to [m]$ in Δ, there is a corresponding simplicial set morphism $g^* \colon \Delta^n \to \Delta^m$ defined by $g^*(l) = g \circ l$ for any k-simplex l.

1.4.1 Application of the Yoneda Lemma and Density Theorem

Let X be a simplicial set. Then the Yoneda Lemma implies that $\mathrm{Hom}(\Delta^n, X) \cong X_n$. There is a natural correspondence between n-simplices of X and simplicial maps $\Delta^n \to X$, and we often abuse notation to use $\sigma \in X_n$ and the corresponding map $\Delta^n \to X$ interchangeably.

We can use the correspondence of n-simplices of X and simplicial maps $\Delta^n \to X$ to depict the n-simplices of X. Let $\sigma \colon \Delta^n \to X$. Then we depict σ in almost the same way we depicted Δ^n. Instead of the vertices being elements of $\Delta^n_0 \cong [n]$ they will instead be elements of $\sigma(\Delta^n_0)$.

Informally, the density theorem shows that every simplicial set is the colimit over all integers $n \geq 0$ of its n-simplices. Given a simplicial set $X \colon \Delta^{op} \to \mathsf{Set}$, we form the category of elements (sometimes called the category of simplices), $\mathcal{I}_X$. The objects of this category are pairs $([n], x)$ where $x \in X_n$, meaning that they correspond to the simplices of X. Given any object $x \in X_n$ and a map $f \colon [m] \to [n]$, there is a corresponding morphism

$(x, f): x \to X(f)(x)$ which acts on x by precomposition. The density theorem implies that we can construct X upto isomorphism as the colimit of $y \circ p$ where p is the forgetful functor $\mathcal{I}_X \to \Delta$, and y is the Yoneda embedding.

In 1.1.1 we gave an alternative but equivalent way to construct to category of objects. Under this definition the objects of this category are simplicial set morphisms $\Delta^n \to X$ (which correspond to the simplices of X).

Given an object $x: \Delta^n \to X$ and a map $f: \Delta^n \to \Delta^m$, there is a corresponding morphism $(x, f): x \to x \circ f$. Note that for any object $x: \Delta^n \to X$ of $\mathcal{I}_X$, $y(p(x)) = \Delta^n$. For notational purposes, we will often not reference $\mathcal{I}_X$ and simply write $X \cong \mathrm{colim}_{\Delta^n \to X} \Delta^n$

Geometric simplicial complexes are constructed by starting with a collection of geometric simplices and the specifying the faces of every geometric simplex, in effect "glues" together the geometric simplices. The density theorem in the context of simplicial sets can be interpreted as saying that simplicial sets are built in a similar way. The colimit can be thought of as taking copies of Δ^n for each n-simplex and "gluing" together any two copies that have a common face.

Given a map of simplicial sets $g: X \to Y$, there is a corresponding functor between their categories of objects $G: \mathcal{I}_X \to \mathcal{I}_Y$, defined as follows. For any object $\sigma: \Delta^n \to X$, let $G(\sigma) = g \circ \sigma$ Given a morphism $(\sigma, f): \sigma \to \sigma \circ f$, where $f: \Delta^m \to \Delta^n$, let $G(\sigma, f) = (G(\sigma), f)$ Then the universal property of the colimit gives a morphism $\mathrm{colim}_{\Delta^n \to X} \Delta^n \to \mathrm{colim}_{\Delta^n \to Y} \Delta^n$.

1.4.2 Faces, Boundaries, and Horns

Definition 1.4.2.1 (Faces and Boundaries of the Standard Simplices)**.** For integers n and i such that $n \geq 0$ and $0 \leq i \leq n$, the ith face of the nth standard simplex, denoted $\delta_i \Delta^n$, has k simplices of the form $f: [k] \to [n]$ such that i is not in the image of f. The boundary of the nth standard simplex, $\delta \Delta^n$ is the union $\cup_{i=0}^n \delta_i \Delta^n$.

Proposition 1.4.2.2. *For any integers n and i such that $0 \leq i \leq n$, $\delta_i \Delta^n$ is a simplicial set. $\delta \Delta^n$ is a simplicial subset for any integer $n \geq 0$.*

Proof. Since $\delta_i \Delta^n$ is a subset of Δ^n, we know that the face and degeneracy maps satisfy the relations (1.2). All that remains to show is that $\delta_i \Delta^n$ is closed under the face and degeneracy maps. This is trivial; if $f\colon [k] \to [n]$ avoids i, then $d_j f = f \circ d^j$ and $s_j f = f \circ s^j$ avoid i for all $0 \leq j \leq n$.

To prove that $\delta\Delta^n$ is a simplicial set, note that the union of simplicial sets is a simplicial set. $\square$

There are two other equivalent ways to define $\delta_i\Delta^n$. First, $\delta_i\Delta^n$ is the image of $d^{i^*}\colon \Delta^{n-1} \to \Delta^n$. Second, $\delta\Delta^n$ is the simplicial subset of Δ^n such that any k-simplex $f\colon [k] \to [n]$ is not surjective. From this we deduce that if $k < n$ then $(\delta\Delta^n)_k = \Delta^n_k$. Similarly, $(\delta\Delta^n)_n = \Delta^n_n \setminus \{id_{[n]}\}$. Thus we can also obtain the boundary of the nth standard simplex from Δ^n by removing the single non-degenerate n simplex $id_{[n]}$, as well as all of its degeneracies.

Δ^n is the combinatorial generalisation of the topological n-ball, and correspondingly, $\delta\Delta^n$ is the generalization of the topological $(n-1)$-sphere. A formal justification for this intuition is given in example Proposition 1.5.0.1.

We can pictorially represent the boundary $\delta\Delta^n$ in the same way as was described for Δ^n, except we imagine that the interior is removed. This prevents one from picking tuples that contains all $n+1$-vertices, which corresponds exactly to the the morphisms of Δ^n that are not present in $\delta\Delta^n$. The ith face, $\delta_i\Delta$, corresponds to the face opposite the ith vertex in this representation.

Definition 1.4.2.3 (Horns)**.** For integers n and i such that $n \geq 0$ and $0 \leq i \leq n$, the ith horn of the nth standard simplex, denoted Λ^n_i, is the simplicial subset of $\delta\Delta^n$ that has k-simplices f such that f avoids i, for all integers $k \geq 0$. A horn is an inner horn if $0 < i < n$. For any simplicial set X, we call a simplicial map $\Lambda^n_i \to X$ an ith n-horn of X.

We can equivalently define the Λ^n_i as the union

$$\bigcup_{j \in [n]\setminus\{i\}} \delta_j\Delta^n.$$

We pictorially represent Λ^n_i in the same way as Δ^n, but with the interior and face opposite the ith vertex removed. This prevents one from forming tuples that contains all vertices, or tuples that contains all except the ith vertex, which correspond precisely to the morphisms of Δ^n that are not present in Λ^n_i.

Note that for $k < n-1$, $(\Lambda^n_i)_k = \Delta^n_k$, and when $k = n-1$, we have that $(\Lambda^n_i)_k = \Delta^n_k \setminus \{d_i\}$. Any horn Λ^n_i is a simplicial set; the proof is nearly identical to that of Proposition 1.4.2.2.

1.5 The Realization of a Simplicial Set

The realization functor $|\cdot|\colon \mathsf{sSet} \to \mathsf{Top}$ gives a way to assign a topological space to any simplicial set. We start by defining the realization of the standard simplices, and using the density theorem, we extend this definition to all simplicial sets. The realization of the standard n-simplex, $|\Delta^n|$, is the standard topological n-simplex, $\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1 \text{ and } x_i \geq 0\}$. This is the geometric n-simplex spanned by the vertices $\{e_i\}_{0 \leq i \leq n}$ where e_i is the ith standard unit vector, and we call e_i the ith vertex of $|\Delta^n|$. For any map $f\colon [n] \to [m]$, we define $|f|\colon |\Delta^m| \to |\Delta^n|$ as $|f|(t_0, \ldots, t_m) = (u_0, \ldots, u_n)$ where $u_i = \sum_{j \in f^{-1}(i)} t_j$. We can define the geometric faces by exploiting the fact that $\delta_i \Delta^n = d^{i^*}(\Delta^{n-1})$. In particular, $|d^{i^*}|$ is the projection of $|\Delta^n|$ onto the face opposite the ith vertex (the geometric simplex spanned by $\{e_j\}_{0 \leq j \leq n, j \neq i}$). With this we have defined the functor $|\cdot|$ for all of the standard simplices.

Recall that for any simplicial set X, $X \cong \operatorname{colim}_{\Delta^n \to X} \Delta^n$. More precisely, X is isomorphic to the colimit of the diagram $y \circ p$. Since $y(p(\sigma)) = \Delta^n$ for any object σ of $\mathcal{I}_X$, $|y(p(\sigma))|$ is already defined, and we define $|X|$ as the colimit of $|y \circ p|$, which we often denote as $\operatorname{colim}_{\Delta^n \to X} |\Delta^n|$. For any simplicial set morphism $g\colon X \to Y$, there is a corresponding functor between their categories of objects, $G\colon \mathcal{I}_X \to \mathcal{I}_Y$. This induces a map $|g|\colon \operatorname{colim}_{\Delta^n \to X} |\Delta^n| \to \operatorname{colim}_{\Delta^n \to Y} |\Delta^n|$. Proving functorialty is trivial since we defined the action of $|\cdot|$ on morphims through the use of induced functors and colimits.

Proposition 1.5.0.1. *$|\Delta^n| \cong D^n$, where D^n denotes the topological n disk and $\delta\Delta^n \cong S^{n-1}$.*

Proof. We will later show that $|\cdot|$ is left adjoint, so it preserves colimits. Since unions are a particular type of colimit, we have the following.

$$
\begin{aligned}
|\delta\Delta^n| &= |\bigcup_{i=0}^{n} \delta_i \Delta| \\
&= |\bigcup_{i=0}^{n} d^{i^*}(\Delta^{n-1})| \\
&= \bigcup_{i=0}^{n} |d^{i^*}(\Delta^{n-1})| \\
&= \bigcup_{i=0}^{n} \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{j=0}^{n} t_j = 1,\, t_j \geq 0 \text{ for all } j \neq i, \text{ and } t_i = 0\} \\
&= \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{j=0}^{n} t_j = 1,\, t_j \geq 0 \text{ for all } j, \text{ and for some } i, t_i = 0\}.
\end{aligned}
$$

$\square$

Proposition 1.5.0.2. *For any simplicial set X the realization $|X|$ can be given the structure of a CW-complex.*

To prove this proposition we will need the machinery of skeletal filtrations of simplicial sets, so we procrastinate on proving this until after Proposition 1.8.0.6.

1.6 Products of Simplicial Sets

The product in $\mathsf{sSet} = \mathsf{Set}^{\Delta^{op}}$ is defined object-wise. That is, if X and Y are simplicial sets then the $(X \times Y)_n = (X \times Y)([n]) = X([n]) \times Y([n]) = X_n \times Y_n$. Verifying that this construction satisfies the universal property is trivial using the universal property of products in Set. For any map $f\colon [n] \to [m]$ the map $(X \times Y)(f)\colon (X \times Y)_m \to (X \times Y)_n$ is defined as $(x, y) \mapsto (X(f)(x), Y(f)(y))$ for any $(x, y) \in (X \times Y)_m$

Proposition 1.6.0.1. *Let X and Y be simplicial sets. Then $|X \times Y| \cong |X| \times |Y|$.*

To prove this one shows that the realization is a left exact and thus preserves finite limits. For a proof consult Theorem 5.2 of [Fri08].

1.7 Internal Hom of sSet

For any set X in Set, the functor $- \times X$ is left adjoint to $\mathrm{Hom}(X, -)$. We want to define a similar internal hom functor $[-, -]\colon \mathsf{sSet}^{op} \times \mathsf{sSet} \to \mathsf{sSet}$ such that for a fixed simplicial set X, $- \times X$ is left adjoint to $[X, -]$. For any sets X and Y, $\mathrm{Hom}(X, Y)$ is itself a set. Similarly, if X and Y are simplicial sets we want for $[X, Y]$ to be a simplicial set.

To construct the internal Hom functor we start by assuming that such a functor with the desired properties exists, and use the Yoneda lemma to arrive at a suitable definition. If the internal Hom functor exists and satisfies the adjointness property discussed, then for any simplicial sets X and Y

$$[X, Y]_n \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, [X, Y]) \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n \times X, Y)$$

Thus we define $[X, Y]$ as the simplicial set where $[X, Y]_n = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n \times X, Y)$ for all integers $n \geq 0$. For any n-simplex $\varphi\colon \Delta^n \times X \to Y$, we define $d_i(\varphi)$ as the composition

$$\Delta^{n-1} \times X \xrightarrow{d^{i*} \times id} \Delta^n \times X \xrightarrow{\varphi} Y,$$

and define $s_i(\varphi)$ as the composition

$$\Delta^{n+1} \times X \xrightarrow{s^{i*} \times id} \Delta^n \times X \xrightarrow{\varphi} Y$$

One can easily check that the face and degeneracy maps satisfy the relations of Equation 1.2, making $[X, Y]$ a simplicial set.

Let $(a^{op}, b)\colon (X, Y) \to (X', Y')$ be a morphism of $\mathsf{sSet}^{op} \times \mathsf{sSet}$. Define $[a^{op}, b]$ as the simplicial morphism sending any $\varphi \in [X, Y]_n$ to the composition $\Delta^n \times X' \xrightarrow{id \times a} \Delta^n \times X \xrightarrow{\varphi} Y \xrightarrow{b} Y'$ (which is an element of $[X', Y']_n$).

Given two simplicial sets X and Y, we define an evaluation map $ev\colon [X, Y] \times X \to Y$ by

$$(f, \sigma) \mapsto f(id_{[n]}, \sigma)$$

where $\sigma\colon \Delta^n \to X$ is an n-simplex of X and $f\colon \Delta^n \times X \to Y$ is an n-simplex of $[X, Y]$.

To prove that the evaluation map is a simplicial morphism, note that it takes n-simplices to n-simplices. So what is left to show is that it commutes with the face and degeneracy maps. We will only show this for the face maps, as the argument for the degeneracy maps is nearly identical. Let $f\colon \Delta^n \times X \to Y$ be an n-simplex of $[X, Y]$, and let $\sigma\colon \Delta^n \to X$

correspond to an n-simplex of X. Then

$$\begin{aligned} ev(d_i(f,\sigma)) &= ev(f \circ (d^i, id_X), d_i\sigma) \\ &= f((d^i, id_X)(id_{[n-1]}, d_i\sigma)) \\ &= f(d^i, d_i\sigma) \\ &= f(d_i(id_{[n]}, \sigma)) \\ &= d_i(f(id_{[n]}, \sigma)) \\ &= d_i(ev(f,\sigma)). \end{aligned}$$

Proposition 1.7.0.1. *For a fixed simplicial set Y, the functors $- \times Y$ and $[Y, -]$ are adjoint.*

Proof. Define a simplicial map $ev_*\colon \mathrm{Hom}_{\mathsf{sSet}}(X, [Y,Z]) \to \mathrm{Hom}_{\mathsf{sSet}}(X \times Y, Z)$ that sends any $g\colon X \to [Y,Z]$ to the composition $X \times Y \xrightarrow{g \times id} [Y,Z] \times Y \xrightarrow{ev} Z$. We must show that ev_* is a bijection that is natural in X and Z. To do this we construct the inverse $\mathrm{Hom}_{\mathsf{sSet}}(X \times Y, Z) \to \mathrm{Hom}_{\mathsf{sSet}}(X, [Y,Z])$, which sends any $f\colon X \times Y \to Z$ to the map f_* where for any $x \in X_n$ (viewed as a map $\Delta^n \to X$), $f_*(x)$ is the composition $Y \times \Delta^n \xrightarrow{id \times x} Y \times X \xrightarrow{g} Z$. Naturality follows from the fact that ev_* is defined by precompositions and postcompositions with morphisms meaning that it automatically commutes with any morphisms. ☐

1.8 Skeletal Filtration and the Dimension of a Simplicial Set

A CW-complex is the union of its skeleta, and one can easily define the dimension of a CW-complex to be the supremum over n such that there exist n-cells. We want to define similar notions for a simplicial set. We will give a recursive way to construct the skeleta of a simplicial set that uses certain pushout diagrams. The image of these pushout diagrams under the realization will define a CW-complex, proving that the realization of a simplicial set is a CW-complex.

The following proposition gives a number of equivalent definitions of a degenerate simplex. The equivalences show that the information conveyed by a degenerate simplices can be conveyed by simplices of a lower dimension, similar to how an n-cell of a given CW-complex may actually be homeomorphic to a disk of dimension less than n.

Proposition 1.8.0.1. *Let S be a simplicial set and let σ be an n-simplex, considered as a map $\sigma\colon \Delta^n \to S$. Then the following are equivalent:*

1. *σ is the image of some degeneracy map $s_i\colon S_{n-1} \to S_n$.*
2. *σ factors as a composition $\Delta^n \xrightarrow{f^*} \Delta^{n-1} \xrightarrow{\sigma'} S$ where f is a surjective map $[n] \to [n-1]$ and σ' is an $(n-1)$-simplex of S.*
3. *σ factors as a composition $\Delta^n \xrightarrow{f^*} \Delta^m \xrightarrow{\sigma'} S$ where $m < n$, $f[n] \to [m]$ is surjective, and σ' is an m-simplex of S.*
4. *σ factors as a composition $\Delta^n \xrightarrow{f^*} \Delta^m \xrightarrow{S}$ where $m < n$ and $f\colon [n] \to [m]$.*

Proof. Clearly $1 \implies 2 \implies 3 \implies 4$, since each is a more general version of the preceding one, so we will show that $4 \implies 1$. Suppose that $m < n$ and σ factors as $\Delta^n \xrightarrow{f^*} \Delta^m \xrightarrow{\sigma'} S$. Since $m < n$, $f\colon [n] \to [m]$ is not injective. Thus for some i, $f(i) = f(i+1)$. Then f factors through the codegeneracy map $s^i\colon [n] \to [n-1]$, and σ factors as the composition $\Delta^n \xrightarrow{s^{i^*}} \Delta^{n-1} \xrightarrow{f'^*} \Delta^m \xrightarrow{\sigma'} S$. Now we simply compose the last two morphisms to obtain σ as the ith degeneracy of $\sigma' \circ f'^*$. □

The following proposition establishes that any degenerate simplex is a degeneracy of a unique non-degenerate simplex. In other words there is no intersection between the degeneracies of two distinct non-degenerate simplices. The proof is lengthy and not particularly insightful, so we have omitted it. It can be found in [Lur06].

Proposition 1.8.0.2. *Let S be a simplicial set and let $\sigma\colon \Delta^n \to S$ be a degenerate n simplex. Then there exists unique m, α, and β such that σ equals the composition*

$$\Delta^n \xrightarrow{\alpha^*} \Delta^m \xrightarrow{\beta^*} S$$

where $m < n$, $\alpha\colon [n] \to [m]$ is surjective and β is a non-degenerate m simplex of S.

Definition 1.8.0.3. Let S be a simplicial set. For each integer $n \geq 0$, $sk_k(S_n)$ is the set of n-simplices of S that factor through Δ^k. The k-skeleton of S is $sk_k(S) := \{sk_k(S_n)\}_{n \geq 0}$.

Recalling Proposition 1.8.0.1, for $n > k$ it is clear that $sk_k(S_n)$ are the n-simplices of S that are degeneracies of m-simplices for $m \leq k$. Thus we can think of $sk_k(S_n)$ as filtering out all of the n-simplices that are not degeneracies of m-simplices for some $m \leq k$. We can understand $sk_k(S)$ as the simplicial subset of S generated by $S_0, S_1, \ldots S_k$. In addition, this proposition shows that the k-skeleton $sk_k(S)$ is closed under the face and degeneracy maps, making it a simplicial set. Since for every $n \geq 0$ if $k \geq n$ then $S_n \subseteq sk_k(S)$, we get that

$$\bigcup_{k \geq 0} sk_k(S) = S. \tag{1.3}$$

This is similar to the construction of CW-complexes as the union of their skeleta. In fact in Proposition 1.8.0.6 we provide a recursive construction of the k-skeleton using pushouts that looks nearly identical to the construction of the skeleta of CW-complexes.

Definition 1.8.0.4. A simplicial set S has dimension $\leq k$ if $n > k$ implies that all of S_n are degenerate simplices. If S has dimension $\leq k$ but not $\leq k-1$ then we say that S has dimension k.

The following proposition asserts that any simplicial morphism from a simplicial set of dimension $\leq k$ is entirely determined its' action m-simplices where $m \leq k$. The proof has been omitted but can be found in [Lur20, Tag 001A].

Proposition 1.8.0.5. *Let S be a simplicial set.*

1. *For any integer $k \geq 0$, $sk_k(S)$ has dimension $\leq k$.*
2. *Let $k \geq 0$ be an integer and let T be a simplicial set such that the dimension of T is $\leq k$. Then composition with the inclusion $sk_k(S) \hookrightarrow S$ induces a bijection*

$$Hom_{\mathsf{sSet}}(T, sk_k(S)) \to Hom_{\mathsf{sSet}}(T, S)$$

Let S be a simplicial set, and denote by S_k^{nd} the non-degenerate k-simplices of S. We can think of any $\sigma \in S_k^{nd}$ as a map $\Delta^k \to sk_k(S)$. Since the dimension of $\delta\Delta^k$ is $\leq k-1$, its image under σ has dimension $\leq k-1$, making it a simplicial subset of $sk_{k-1}(S)$. Thus for any $\sigma \in S_k^{nd}$ there is a corresponding map $\delta\Delta^k \to sk_{k-1}(S)$.

Proposition 1.8.0.6. *The above construction determines a pushout square for all integers $k \geq 0$, depicted below.*

$$\begin{tikzcd}
\coprod_{\sigma \in S_k^{nd}} \delta\Delta^k \arrow[r, hook] \arrow[d] & \coprod_{\sigma \in S_k^{nd}} \Delta^k \arrow[d] \\
sk_{k-1}(S) \arrow[r, hook] & sk_k(S)
\end{tikzcd}$$

A proof of this proposition can be found in [Lur20, Tag 001B]

This proposition makes it easy to prove Proposition 1.5.0.2, which states that the geometric realization of any simplical set can be given the structure of a CW-complex.

Proof. (of Proposition 1.5.0.2). In Proposition 2.3.0.3 we show that the geometric realization functor is left adjoint. Left adjoints commute with colimits, and in particular the realization commutes with pushouts. Thus by applying the realization functor to the pushout diagram of Proposition 1.8.0.6, we get the following pushout diagram.

$$\begin{tikzcd}
|\coprod_{\sigma\in S_k^{nd}} \delta\Delta^k| \cong \coprod_{\sigma\in S_k^{nd}} |\delta\Delta^k| \arrow[r, hook] \arrow[d] & |\coprod_{\sigma\in S_k^{nd}} \Delta^k| \cong \coprod_{\sigma\in S_k^{nd}} |\Delta^k| \arrow[d] \\
|sk_{k-1}(S)| \arrow[r, hook] & |sk_k(S)|
\end{tikzcd}$$

Note that by Proposition 1.5.0.1 this pushout is effectively gluing k-disks to $|sk_{k-1}(S)|$ along their boundaries. Now if we show that $|S| = \bigcup_{k\geq 0} |sk_k(S)|$, then by definition this gives $|S|$ the structure of a CW-complex. Note that we can obtain the union $\bigcup_{k\geq 0} sk_k(S)$ as the colimit of the diagram

$$sk_0(S) \hookrightarrow sk_1(S) \hookrightarrow sk_2(S) \hookrightarrow \cdots .$$

Thus Equation (1.3) and the left adjointness of $|\cdot|$ proves the result. $\square$

Chapter 2

The Nerve and Singular Functors

In this chapter we introduce the nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$, and the singular functor $\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{sSet}$. These functors will help motivate the model of ∞-categories presented in Chapter 3.

The nerve of a category and the singular set of a space are important examples of ∞-categories. Recall that our goal is to present ∞-categories as categories in which the notion of equality has been replace with isomorphism. Topological homotopy theory already has a notion of isomorphism, namely homotopy. In Chapter 3 we will study what happens to homotopies under the singular functor. Generalizing the properties of homotopies under the singular functor will guide our search for the correct definition of isomorphism in ∞-categories (which we will refer to as homotopy). The nerve is fully faithful, and considering the action of the nerve functor on various categorical constructions motivate corresponding constructions in the world of ∞-categories where we can no longer use the notion of equality.

2.1 The Nerve of a Category

Definition 2.1.0.1. The nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$ is defined in the following way:

- For any small category $\mathcal{C}$, $N(\mathcal{C})$ is the simplicial set where $N(\mathcal{C})_n$ is the set of functors $[n] \to \mathcal{C}$. For any morphism $f\colon [n] \to [m]$ of Δ, let $N(\mathcal{C})(f)\colon N(\mathcal{C})_m \to N(\mathcal{C})_n$ be defined by precomposition with f.
- If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, then we define $N(F)$ by postcomposition with F.

Checking that $N(\mathcal{C})$ is a simplicial set for any small category $\mathcal{C}$ amounts to checking that the relations (1.2) are satisfied, which is trivial.

Remark 2.1.0.2. The 0-simplices of $N(\mathcal{C})$ can be identified with objects of $\mathcal{C}$, and the 1-simplices can be identified with morphism. For any $n \geq 1$, an n-simplicx σ of $N(\mathcal{C})$ can be identified with strings of n composable morphisms of $\mathcal{C}$

$$(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n). \tag{2.1}$$

We will gleefully and repeatedly abuse notation by using an object c and the corresponding 0-simplex of $N(\mathcal{C})$ interchangeably. Similarly, for any morphism f of $\mathcal{C}$ we will use it and the corresponding 1-simplex of $N(\mathcal{C})$ interchangeably.

Given such an identification of σ as in (2.1), $d_i(\sigma)$ can be identified with $(C_1 \xrightarrow{f_1} \cdots C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} C_n)$ for $0 < i < n$. If $i = 0$ or $i = n$, $d_i\sigma$ is obtained by deleting C_0 and f_1 from σ, or deleting C_n and f_n from σ, respectively. Similarly, $s_i\sigma = \sigma s^i$ can be identified with $(C_0 \xrightarrow{f_0} \cdots \xrightarrow{f_i} C_i \xrightarrow{id_i} C_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} C_n)$. For any object c of $\mathcal{C}$, the morphism corresponding to id_c is $s_0(c)$. For any 2-simplex τ of $N(\mathcal{C})$, we have that $d_1(\tau) = d_0(\tau) \circ d_2(\tau)$. An n-simplex of $N(\mathcal{C})$ is non-degenerate precisely when none of the morphisms in the associated string are identity morphisms.

Proposition 2.1.0.3. *The nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We want to show that the nerve functor gives a bijection $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{sSet}}(N(\mathcal{C}), N(\mathcal{D}))$.

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be distinct functors. Then for some morphism ρ of $\mathcal{C}$, $F(\rho) \neq G(\rho)$. We continue the tradition of abusing notation by considering ρ as a 1-simplex of $N(\mathcal{C})$. In particular, this means that $N(F)(\rho) = F(\rho) \neq G(\rho) = N(G)(\rho)$. Thus $N(F) \neq N(G)$, and N gives an injective map $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{sSet}}(N(\mathcal{C}), N(\mathcal{D}))$

Now let $f\colon N(\mathcal{C}) \to N(\mathcal{D})$ be a simplicial set morphism. By Remark 2.1.0.2 any n-simplex of $N(\mathcal{C})$ is determined by its 1-faces. Thus any simplicial set morphism from $N(\mathcal{C})$ is determined by its action on 0 and 1-simplices. In particular, f is determined by its action on the 0 and 1-simplices of $N(\mathcal{C})$, which correspond to objects and morphisms of $\mathcal{C}$, respectively. Thus there is a functor $\tilde{F}\colon \mathcal{C} \to \mathcal{D}$ corresponding to f. Clearly for 0 and 1-simplices $N(\tilde{F})$ and f agree, and by the observation that f is determined by its action on the 0 and 1-simplices of $N(\mathcal{C})$, we have that $N(\tilde{F}) = f$. Thus N gives a surjective map $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{sSet}}(N(\mathcal{C}), N(\mathcal{D}))$. $\square$

2.1.1 Extension Properties

Proposition 2.1.1.1. *For any small category $\mathcal{C}$, $N(\mathcal{C})$ satisfies the following condition: For any integer $n \geq 2$, integer i such that $0 < i < n$, and simplicial morphism $\sigma_0 \colon \Lambda^n_i \to N(\mathcal{C})$ there is a unique extension of σ_0 to Δ^n.*

Proposition 2.1.1.1 is obvious when $n = 2$ (which forces i=1). In this case, a simplicial set morphism $\Lambda^2_1 \to N(\mathcal{C})$ is equivalent to giving two composable morphisms of $\mathcal{C}$. We want to extend this to a 2-simplex, which we can do by defining a functor $[2] \to \mathcal{C}$ which the morphism $0 \leq 2$ in $[n]$ to the composition of the two morphisms

$$\begin{tikzcd} & C_1 \arrow[dr, "f_{2,1}"] & \\ C_0 \arrow[ur, "f_{1,0}"] & & C_2 \end{tikzcd} \qquad \begin{tikzcd} & C_1 \arrow[dr, "f_{2,1}"] & \\ C_0 \arrow[ur, "f_{1,0}"] \arrow[rr, "f_{2,1} \circ f_{1,0}"] & & C_2 \end{tikzcd}$$

Figure 2.1: The first figure is the image of Λ^2_1, and the second is the extension to Δ^2. Note that these do not depict commutative diagrams within $\mathcal{C}$, they are only collections of 0 and 1-simplices of $N(\mathcal{C})$.

More generally, Proposition 2.1.1.1 follows from the uniqueness of composition in categories.

Proof. Pick any small category $\mathcal{C}$, integer $n > 0$, and $0 < i < n$, and let $\sigma_0 \colon \Lambda^n_i \to N(\mathcal{C})$ be a map of simplicial sets. Define C_j to be the object of $\mathcal{C}$ corresponding to the image of the jth vertex of Δ^n under σ_0.

First we consider the case of when $n \geq 3$. In this case all of the 1-simplices of Δ^n are contained in Λ^n_i. Then for integers $0 \leq j \leq k \leq n$, let $f_{k,j}$ be the image under σ_0 of the 1-simplex $(j \leq k)$ of Δ. We claim that the map induced by $j \mapsto C_j$ and $(j \leq k) \mapsto f_{k,j}$ is a functor $[n] \to \mathcal{C}$. The induced map takes an m-simplex $(i_0 \leq \cdots \leq i_m)$ of Δ^n to the diagram depicted by

$$(C_0 \xrightarrow{f_{i_1,i_0}} C_1 \xrightarrow{f_{i_2,i_1}} \cdots \xrightarrow{f_{i_m,i_{m-1}}} C_m).$$

To prove functoriality (and well definedness) it is obvious that identity maps are taken to identity maps, and we must show that for any $0 \leq j \leq k \leq l \leq n$, $f_{l,j} = f_{l,k} \circ f_{k,j}$. If the two simplex τ corresponding to $(j \leq k \leq l)$ is in Λ^n_i, then $\tau' \colon = \sigma_0(\tau)$ is a 2-simplex of $N(\mathcal{C})$, and since $d_1(\tau') = f_{l,j}$, $d_0(\tau') = f_{l,k}$, and $d_1(\tau') = f_{k,j}$, we have that $f_{l,j} = f_{l,k} \circ f_{k,j}$.

Now suppose that τ is not contained in Λ_i^n. Since for $m < n-1$, $(\Lambda_i^n)_m = \Delta_m^n$, it must the be the case that $2 < n-1$, and by the assumption that $n \geq 3$, we get that $n = 3$. Thus i is either 1 or 2. We will only prove the case of $i = 1$, as the case of $i = 2$ is very similar. Since Λ_1^3 contains all of the 2-simplices of Δ^n other than one, this missing one must be τ. Thus we get that $(j \leq k \leq l) = (0 \leq 2 \leq 3)$, and we want to show that $f_{3,0} = f_{3,2} \circ f_{2,0}$. By applying σ_0 to the 2-simplices of Λ_1^3 that are in Δ^n, we get the 2-simplices of $N(\mathcal{C})$ that "witness" the following

$$f_{3,1} = f_{3,2} \circ f_{2,1} \qquad f_{3,0} = f_{3,1} \circ f1,0 \qquad f_{2,1} = f_{2,1} \circ f_{1,0}.$$

We put all of this together to compute that

$$f_{3,0} = f_{3,1} \circ f_{1,0} = f_{3,2} \circ f_{2,1} \circ f_{1,0} = f_{3,2} \circ f_{2,0}.$$

Thus we have a map $[n] \to \mathcal{C}$, which by definition is an n-simplex of $N(\mathcal{C})$, and this is equivalent to giving a map $\Delta^n \to N(\mathcal{C})$. By construction this extends σ_0, and since there were no choices made in the construction, this extension is unique.

All that remains is the case of $n = 2$ in this case $i = 1$. Then any $\sigma_0 \colon \Lambda_1^2 \to N(\mathcal{C})$ specifies a pair of composable morphisms f and g, and the functor $[n] \to \mathcal{C}$ given by $(0 \leq 1) \mapsto f$, $(1 \leq 2) \mapsto g$, $(0 \leq 2) \mapsto g \circ f$ is an n-simplex of $\mathcal{C}$, (which is equivalent to a map $\Delta^n \to N(\mathcal{C})$) that extends σ_0. Uniqueness follows from the uniqueness of composition in categories. $\square$

2.2 The Singular Functor

Definition 2.2.0.1. The singular functor $\mathrm{Sing} \colon \mathsf{Top} \to \mathsf{sSet}$ is defined in the following way.

- For any space X let $\mathrm{Sing}(X)$ be the simplicial set where $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, X)$. The face map $d_i \colon \mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n-1}$ is defined by precomposition; for any n-simplex $\varphi \colon |\Delta^n| \to X$, $d_i(\varphi)$ is the composition $|\Delta^{n-1}| \xrightarrow{|d^{i*}|} |\Delta^n| \xrightarrow{\varphi} X$. The degeneracy maps are defined by precomposition in the same way.
- For any morphism $f \colon X \to Y$ in Top, $\mathrm{Sing}(f)$ is defined by composition with f.

Since $|\Delta^0| \cong \{*\}$ and $|\Delta^n| \cong [0,1]$ (where $[0,1]$ denotes the unit interval), the 0-simplices of $\mathrm{Sing}(X)$ can be identified with the points of X, and the 1-simplices can be identified with paths $[0,1] \to X$.

Proposition 2.2.0.2. *Sing is a functor* $\mathsf{Top} \to \mathsf{sSet}$.

Proof. First we must show that given any space X in Top, $\mathrm{Sing}(X)$ is a simplicial set. Recall that we can do this by showing that the Relations (1.2) hold, which is trivial. Next we must show that for any continuous map $f\colon X \to Y$ in Top, f induces a simplicial map $\mathrm{Sing}(X) \to \mathrm{Sing}(Y)$. $\mathrm{Sing}(f)$ takes any n-simplex $\varphi\colon |\Delta^n| \to X$ to the composition $f \circ \varphi$. Clearly this takes n-simplices of $\mathrm{Sing}(X)$ to n-simplices of $\mathrm{Sing}(Y)$, but remains to check is that it commutes with the face and degeneracy maps. We will show the proof for face maps and the proof for degeneracy maps is nearly identical. $\mathrm{Sing}(f)(d_i(\varphi)) = \mathrm{Sing}(f)(\varphi \circ |d^i|) = f \circ \varphi \circ |d^i| = d_i(f \circ \varphi) = d_i(\mathrm{Sing}(f)(\varphi))$. Finally, we note that Sing respects composition of morphisms by definition. □

2.2.1 Kan Complexes

Like the nerve of a small category, the singular set of a topological space has a similar extension property.

Definition 2.2.1.1. A simplicial set X is a Kan Complex if for every $0 \leq i \leq n$ and simplicial map $\Lambda^n_i \to X$, there is an extension $\Delta^n \to X$.

Unlike the extension property for the nerve of a category (Proposition 2.1.1.1), a Kan complex requires extensions when $i = 0$ or $i = n$, but it does not require that these extensions are unique.

Example 2.2.1.2. Δ^n is not a Kan complex for $n > 0$. For a counter-example, consider the $\sigma_0\colon \Lambda^2_0 \to \Delta^2$ whose action on 0-simplices sends 0 to 0, 1 to 2, and 2 to 1, and whose action 1-simplices sends the $(0 \leq 2)$ in Λ^2_0 to $0 \leq 1$ in Δ^2, and sends $0 \leq 1$ to $0 \leq 2$. Then any extension of σ_0 to all of Δ^2 would require sending the 1-simplex $1 \leq 2$ to a morphism from 2 to 1, which is impossible as there are no such morphisms.

Proposition 2.2.1.3. *For any space X in* Top*, $\mathrm{Sing}(X)$ is a Kan Complex.*

Proof. Let X be an object of Top, $0 \leq i \leq n$, and let $\sigma_0\colon \Lambda^n_i \to \mathrm{Sing}(X)$ be a simplicial map, we want to show that there is a simplicial map $\sigma\colon \Delta^n \to \mathrm{Sing}(X)$ that extends σ_0.

Figure 2.2: On the left is a depiction of Λ^2_0 and on the right is a depiction of its image under σ_0. The dashed arrow corresponds to the single 1-simplex of Δ^2 that is not in Λ^2_0. An extension of σ_0 to all of Δ^2 would would send the dashed arrow to an arrow from 2 to 1.

Note that such a σ is equivalent to an n-simplex of $\mathrm{Sing}(X)$ by the Yoneda Lemma. We will later prove Proposition 2.3.0.3, which shows that the realization and singular functors are adjoint, which allows us to identify σ_0 with a continuous map $f\colon |\Lambda^n_i| \to X$. Since there is a deformation retraction $r\colon |\Delta^n| \to |\Lambda^n_i|$, we consider the composition $|\Delta^n| \xrightarrow{r} |\Lambda^n_i| \xrightarrow{f} X$, which by adjointness can be identified with a map $\sigma\colon \Delta^n \to \mathrm{Sing}(X)$. To prove that this is the desired extension, we must show that the composition $\Lambda^n_i \hookrightarrow \Delta^n \to \mathrm{Sing}(X)$ equals σ_0. By adjointness, this map corresponds to $|\Lambda^n_i| \hookrightarrow |\Delta^n| \xrightarrow{f\circ r} X$. This composition must be f (since r fixes elements of $|\Lambda^n_i|$), using adjointness again, we get σ_0. $\square$

Remark 2.2.1.4. For any space X, $|\mathrm{Sing}(X)|$ is weak homotopy equivalent to X. In fact, we can view $|\mathrm{Sing}(X)|$ as a CW approximation of X (see Proposition 1.5.0.2 for more details). If $f\colon X \to Y$ is a weak homotopy equivalence of spaces, then $|\mathrm{Sing}(f)|$ is as well. We call a simplicial map between Kan complexes g a weak equivalence if $|g|$ is a weak homotopy equivalence of spaces. Any Kan complex is weakly equivalent to the singular set of some space. These properties taken together show that the study of Kan complexes and weak homotopy equivalences between them encapsulates the homotopy theory of spaces. Proofs of all of these statements can be found in [Lur06].

2.3 A Unified Treatment

The nerve and singular functors are specific examples of a broader class of functors. Since s simplicial set is isomorphic to the colimit of all of its simplicies, we can often uniquely describe functors from sSet to another category by specifying its action on the standard simplicies.

Let $\mathcal{D}$ be any cocomplete category, and let $F\colon \Delta \to \mathcal{D}$ be a functor. The goal is to use F

to construct a pair of adjoint functors $\mathsf{sSet} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$.

$$\begin{tikzcd}
 & & \Delta \arrow[ddll, hook', "y"'] \arrow[ddrr, "F"] & & \\
 & & n \arrow[dl, mapsto] \arrow[dr, mapsto] & & \\
\mathsf{sSet} \arrow[rrrr, shift left, "L"] & \Delta^n \arrow[rr, mapsto] & & F([n]) & \mathcal{D} \arrow[llll, shift left, "R"]
\end{tikzcd}$$

Let $y\colon \Delta \to \mathsf{sSet}$ be the Yoneda embedding. We define $L'(\Delta^n) = L'(y([n]))$ to be equal to $F([n])$. We can extend L' to all of sSet by requiring that it commutes with colimits. Let X be a simplicial set Since $X \cong \mathrm{colim}(y \circ p)$, we define $L(X)$ as $\mathrm{colim}(L' \circ y \circ p)$. For notational purposes, we write

$$L(X) \cong L(\mathrm{colim}_{\Delta^n \to X} \Delta^n) = \mathrm{colim}_{\Delta^n \to X}(F([n]).$$

Any simplicial morphism $g\colon X \to Y$ induces a functor between their categories of objects $G\colon \mathcal{I}_X \to \mathcal{I}_Y$. G then induces a a unique map $L(g)\colon L(X) \to L(Y)$.

If one thinks of F as giving a realization of the standard simplices in the category $\mathcal{D}$, then L extends this realization to all simplicial sets by using the density theorem. This is exactly what was done when constructing the geometric realization functor, we first defined the realization in Top of the standard simplices, and then extended this to a realization in Top for any simplicial set.

The construction of $R\colon \mathcal{D} \to \mathsf{sSet}$ follows a similar thought process. Given an object D of $\mathcal{D}$, we want to construct a simplicial set $R(D)$. Up until now, our application of the density theorem involved deconstructing a given simplicial set as the colimit of its simplices. Now we work in reverse; if we think of $F([n])$ as the realization of Δ^n in $\mathcal{D}$, then it would be natural to set $R(D)_n \colon = \mathrm{Hom}_{\mathcal{D}}(F([n]), D)$.

We define the face and degeneracy maps via precomposition as follows. Let φ be an n-simplex of $R(D)$. Then

- $d_i(\varphi) = \varphi \circ F(d^i)$, and
- $s_i(\varphi) = \varphi \circ F(s^i)$.

It is easy to check that this satisfies the relations of Equation (1.2), making $R(D)$ a simplicial set. Given a morphism $f\colon D \to E$ in $\mathcal{D}$, we define $R(f)$ by composition with f. More precisely, $R(f)$ sends any n-simplex φ to $f \circ \varphi$.

Proposition 2.3.0.1. *L is left adjoint to R.*

Proof. Let D be an object of $\mathcal{D}$ and $n \geq 0$. By the Yoneda Lemma and by definition

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, R(D)) \cong R(D)_n = \mathrm{Hom}_{\mathcal{D}}(F([n]), D) = \mathrm{Hom}_{\mathcal{D}}(L(\Delta^n), D).$$

Now let X be any simplicial set and D be any object of D. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{sSet}}(X, R(D)) &\cong \mathrm{Hom}_{\mathsf{sSet}}(\mathrm{colim}_{\Delta^n \to X} \Delta^n, R(D)) \\
&\cong \lim_{\Delta^n \to X} \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, R(D)) \\
&\cong \lim_{\Delta^n \to X} R(D)_n = \lim_{\Delta^n \to X} \mathrm{Hom}_{\mathcal{D}}(L(\Delta^n), D) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}_{\Delta^n \to X} L(\Delta^n), D) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(X), D).
\end{aligned}$$

□

Example 2.3.0.2. Let $\mathcal{D} = \mathsf{Cat}$, and let $F\colon \Delta \to \mathsf{Cat}$ be the inclusion. Then $R\colon \mathsf{Cat} \to \mathsf{sSet}$ sends a category $\mathcal{C}$ to the simplicial set $R(\mathcal{C})$ where $R(\mathcal{C})_n = \mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C}) = N(\mathcal{C})_n$, so $R = N$. In particular, the nerve functor is right adjoint.

On the other hand, $L\colon \mathsf{sSet} \to \mathsf{Cat}$ sends any simplicial set X to the category $\mathrm{colim}_{\Delta^n \to X}[n]$, where $[n]$ is regarded as a category. L in this case essentially "forgets" the structure of X for any dimension > 2. We construct $L(S)$ explicitly in the following way. For any 0-simplex x of S, let $\overline{x}$ be the corresponding object in $L(S)$. Then for any 1-simplex f such that $d_0(f) = b$ and $d_1(f) = a$, let there be a corresponding morphism $\overline{f}\colon \overline{a} \to \overline{b}$. For any 1-simplex x, we define the identity morphism of $\overline{x}$ as $\overline{s_0(x)}$. We begin by defining composition of such morphisms freely and formally, and then impose the following restriction: $\overline{f} \circ \overline{g} = \overline{h}$ exactly when there is a 2-simplex σ such that $d_0(\sigma) = f$, $d_1(\sigma) = h$ and $d_2(\sigma) = g$.

$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ x \arrow[ur, "g"] \arrow[rr, "h"] & & z \end{tikzcd} \qquad \begin{tikzcd} & \overline{y} \arrow[dr, "\overline{f}"] & \\ \overline{x} \arrow[ur, "\overline{g}"] \arrow[rr, "\overline{h}"] & & \overline{z} \end{tikzcd}$$

Figure 2.3: A 2-simplex of X (left) and the corresponding commutative diagram in $L(X)$ (right).

Proposition 2.3.0.3. *Let $F\colon \Delta \to \mathsf{Top}$ be the functor sending $[n]$ to $|\Delta^n|$. Then in the construction above, $|\cdot| = L\colon \mathsf{sSet} \to \mathsf{Top}$ and $Sing = R\colon \mathsf{Top} \to \mathsf{sSet}$. In particular, the singular and realization functors are adjoint.*

The proof is a trivial application of the definitions.

Chapter 3

∞-Categories

Recall that we want to think of ∞-categories as categories in which the notion of equality has been replaced by isomorphism (which we call homotopy). Here, we make this idea rigorous by introducing a model for ∞-categories as well as several key constructions and proofs.

Recall the extension properties of the nerve and singular functors. ∞-categories generalize both category theory and the homotopy theory of spaces by introducing a common generalization of these extension properties.

Definition 3.0.0.1. A simplicial set X is an ∞-category if for all integers $0 < i < n$, and simplicial morphisms $\Lambda^n_i \to X$, there is an extension $\Delta^n \to X$.

Remark 3.0.0.2. Clearly both the nerve of a small category and the simplicial set of a space are ∞-categories.

Remark 3.0.0.3. Other sources may refer to this model of ∞-categories as "quasi-categories".

For any ∞-category $\mathcal{D}$, we call simplicial maps $h\colon \Lambda^n_i \to \mathcal{D}$ "horns in $\mathcal{D}$". Since $\Lambda^n_i = \cup_{j\in[n]\setminus\{i\}}\delta_j\Delta^n$, a horn h in $\mathcal{D}$ is equivalent to the collection $\{h|_{\delta_j\Delta^n}\}_{j\in[n]\setminus\{i\}}$. The map $d^{j^*}\colon \Delta^{n-1} \to \Delta^n$ sends Δ^{n-1} to $\delta_i\Delta^n$ in a bijective way, so we can rewrite our collection $\{h_j = h\circ d^{j^*}\colon \Delta^{n-1} \to \mathcal{D}\}_{j\in[n]\setminus\{i\}}$. Thus we can specify any horn by a particular collection of n simplices of dimension $n-1$. By viewing Λ^n_i as a subsimplicial set of Δ^n and considering the relations (1.2), the collection of maps $\{h_i\}$ have the property that $d_k(h_j) = d_{j-1}(h_k)$ for $k < j$.

In fact, for any collection of maps (or $(n-1)$-simplices) $\{f_j\colon \Delta^{n-1} \to \mathcal{D}\}_{j\in[n]\setminus\{i\}}$ such that $d_k(f_j) = d_{j-1}(f_k)$ when $k < j$, we can reverse the above arguments to specify a horn $\Lambda^n_i \to \mathcal{C}$. We will often specify such a set by a tuple $(f_0, \cdots, *, \cdots, f_n)$ where the entry $*$ is in the ith position.

Let $\mathcal{C}$ be an ordinary category. Then the 0-simplices of $N(\mathcal{C})$ correspond to the morphisms of $\mathcal{C}$ and the 1-simplices correspond to objects of $\mathcal{C}$. We generalize this in the most straight forward way possible.

Definition 3.0.0.4. Let $\mathcal{D}$ be an infinity category. Then we call the 0-simplices of $\mathcal{D}$ objects and the 1-simplices morphisms. Given a 1-simplex f such that $d_1(f) = x$ and $d_0(f) = y$, we write $f\colon x \to y$. Given an object d of some ∞-category $\mathcal{D}$, we call $s_0(c)$ the identity morphism of d, denoted id_d.

If X is a space in Top, then objects of $\mathrm{Sing}(X)$ correspond to points of X, while morphisms correspond continuous paths $f\colon [0,1] \cong |\Delta^1| \to X$, which we denote as $f\colon f(0) \to f(1)$.

Remark 3.0.0.5. We will generally denote the objects of an ∞-category using capital letters in contexts related to the nerve functor, as objects of categories are often denoted with capital letters. We will generally denote objects of ∞-categories using lower case letters in contexts relating to the singular functor, as points of a space are usually denoted with lower case letters.

3.1 Homotopies of Morphisms

In order to get rid of the notion of equality of morphisms we must define a notion of homotopies of morphisms. We already have a topological notion of homotopy and we will use this along with the singular functor to motivate our general definition of homotopies of morphisms.

Let X be an object of Top. Morphisms of $\mathrm{Sing}(X)$ correspond to paths $[0,1] \to X$. Let f and g be morphisms such that $f(0) = g(0) = x$ and $f(1) = g(1) = y$. Then a topological (fixed endpoint) homotopy from f to g is a map $H\colon [0,1]\times[0,1] \to X$ such that $H(0,t) = x$, $H(1,t) = y$, $H(s,0) = f(s)$, and $H(s,1) = g(s)$. We form the quotient space $Q := ([0,1]\times[0,1])/(\{0\}\times[0,1])$, and since H is constant along the side $\{0\}\times[0,1]$, $H'\colon Q \to X$ where $[p] \mapsto H(p)$ is well defined and continuous.

Figure 3.1: A depiction of H (left) and H' (right)

The map $p\colon Q \to |\Delta^2|$ defined by $[(x,y)] \mapsto (1-x-y, x, y)$ is a homeomorphism. Consider $H' \circ p^{-1}\colon |\Delta^2| \to X$. By definition $\sigma := H' \circ p^{-1}$ is a 2-simplex of $\mathrm{Sing}(X)$. $d_2(\sigma) = H' \circ p^{-1} \circ |d^{2^*}| = f\colon [0,1] \to X$, $d_1(\sigma) = g$, and $d_0(\sigma) = y$, when viewing $|\Delta^1|$ as $[0,1]$. Note that after picking a homotopy H, no choices were made in the construction of σ, and in fact we can reverse the construction to recover H from σ.

Thus given two 1-simplices $f, g\colon x \to y$ in $\mathrm{Sing}(X)$, we define a homotopy from f to g is a 2-simplex σ such that $d_0(\sigma) = id_y$, $d_1(\sigma) = g$ and $d_2(\sigma) = f$. By the above arguments, homotopies from f to g in $\mathrm{Sing}(X)$ correspond bijectively to topological fixed-point homotopies from f to g in X. We accept this definition of a homotopy of morphisms with out any modifications in a general ∞-category.

Definition 3.1.0.1. Let $\mathcal{D}$ be an ∞-category. Then given two morphisms $f, g\colon x \to y$ of $\mathcal{D}$, a homotopy from f to g is a 2-simplex σ such that $d_0(\sigma) = id_y$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$.

$$\begin{tikzcd} & D \arrow[dr, "id_y"] & \\ C \arrow[ur, "f"] \arrow[rr, "g"] & & D \end{tikzcd}$$

Figure 3.2: A depiction of a homotopy from f to g.

We will try to always make reference to a specific homotopy from f to g rather that asserting that they are homotopic. This emphasizes that we have replaced the abstract notion of equality with the concrete structure of homotopies.

The relation induced by fixed point homotopies is an equivalence relation on the set paths from x to y. The correspondence between homotopies of morphisms in $\mathrm{Sing}(X)$ and topological homotopies of paths in X shows that homotopies of morphisms in $\mathrm{Sing}(X)$ induce an equivalence relation on the set of morphisms from x to y. The following proposition shows that this is true in general, and is the first proof that takes advantage of the horn filling condition of ∞-categories.

Proposition 3.1.0.2. *Let $\mathcal{D}$ be an ∞-category, let x and y be objects of $\mathcal{D}$, and denote by E the set of morphisms from x to y. Then homotopy is an equivalence relation on E.*

Proof. To show reflexivity, let $f\colon X \to Y$ be a morphism. Then $s_1(f)$ is a homotopy from f to itself.

Now let f, g, and h be morphisms $X \to Y$ such that there exists a homotopy σ_2 from f to h, and a homotopy σ_3 from f to g. Then we can construct a simplicial map $\Lambda^3_1 \to \mathcal{D}$ corresponding to the tuple $(s_1(s_0(Y)), *, \sigma_2, \sigma_3)$, which is depicted in the following diagram.

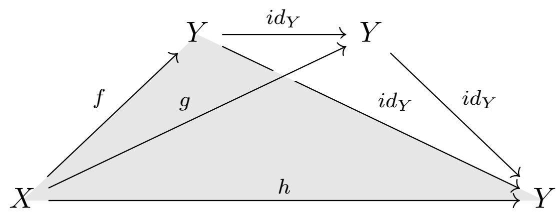

Figure 3.3: The shaded portion represents the 2-simplex that must be specified to extend this horn to a 3-simplex.

There must be an extension of this map to Δ^n. In particular, the image of $\delta_3\Delta$ under this extension (corresponding to the shaded face in the diagram) will be a homotopy from f to g.

By letting $h = f$ the above construction shows symmetry of homotopy. Symmetry shows that the statements "f is homotopic to g and f is homotopic to h" and "g is homotopic to f and f is homotopic to g" are equivalent. Then the previous argument shows the transitivity of homotopy. $\square$

The following proposition will be used in many proofs.

Proposition 3.1.0.3. *Let $\mathcal{D}$ be an ∞-category, and let $f, f'\colon x \to y$. Then the following conditions are equivalent:*

1. *There exists a 2-simplex σ of $\mathcal{D}$ such that $d_0(\sigma) = id_y$, $d_1(\sigma) = f'$, and $d_2(\sigma) = f$.*

2. *There exists a 2-simplex τ of $\mathcal{D}$ such that $d_0(\tau) = f'$, $d_1(\tau) = f$, and $d_2(\tau) = id_C$.*

$$\begin{tikzcd} & y \arrow[dr, "id_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f'"] & & y \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f'"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"] & & y \end{tikzcd}$$

Figure 3.4: A simplex satisfying Condition 1 (left), and a simplex satisfying Condition 2 (right).

This proposition essentially gives an equivalent definition of a homotopy between two morphisms. We would have gotten this definition if we had chosen to collapse the side $\{1\} \times [0, 1]$ instead of $\{0\} \times [0, 1]$ in the construction at the beginning of this section.

Proof. Suppose that there exists a 2-simplex σ satisfying Condition 1. Consider the horn $\gamma_0 \colon \Lambda^3_2 \to \mathcal{C}$ corresponding to the tuple $(\sigma, s_1(f), *, s_0(f))$, depicted below.

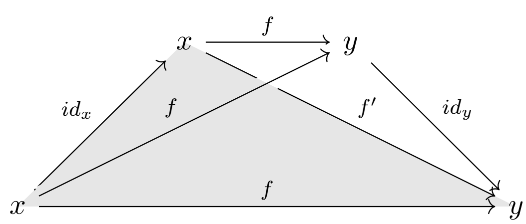

Figure 3.5: The shaded portion represents the 2-simplex that must be specified to extend this horn to a 3-simplex.

There is an extension $\gamma \colon \Delta^3 \to \mathcal{D}$, and $\tau := \gamma(\delta_2(\Delta^n))$ satisfies Condition 2. There is a similar proof of the converse statement. $\square$

3.2 Compositions of Morphisms

In ordinary categories the composition of two morphisms is unique. Another way to phrase this is that for any composable morphisms f and g, any two compositions of f and g are equal. As we have been stressing to the point of annoyance, we are no longer allowed to use the concept of equality in the setting of ∞-categories. Thus we would want for there to be a homotopy between any two compositions of f and g, where f and g are composable morphisms in an ∞-category.

We begin by considering the composition of morphisms in an ordinary category $\mathcal{C}$ and what the corresponding structure is in its nerve, $N(\mathcal{C})$. Given two composable morphisms

$f\colon x \to y$ and $g\colon y \to z$ in $\mathcal{C}$, we can form the horn $\Lambda^2_1 \to N(\mathcal{C})$ corresponding to $(g, *, f)$, and by the definition of ∞-categories, there is a filling of this horn to a map $\Delta^n \to N(\mathcal{C})$, which corresponds to an n-simplex $\sigma\colon [n] \to \mathcal{C}$ of $N(\mathcal{C})$. Recall that such extensions are unique when working with the nerve of a category. Note that σ then must be the only 2-simplex such that $d_0(\sigma) = g$, $d_2(\sigma) = f$, and $d_1(\sigma) = d_1(\sigma) \circ d_2(\sigma) = g \circ f$. We can say that the 2-simplex σ "witnesses" $g \circ f$ as a composition of g and f, where f, g, and $g \circ f$ are considered as morphisms of $N(\mathcal{C})$. We generalize this definition in the following way.

Definition 3.2.0.1. Given an ∞-category $\mathcal{D}$ and morphisms $f\colon x \to y$ and $g\colon y \to z$, we say that $h\colon x \to z$ is a composition of g and f if there exists a 2-simplex σ such that $d_0(\sigma) = g$, $d_1(\sigma) = h$, and $d_2(\sigma) = f$. In this case we say that σ witnesses h as a composition of g and f.

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

Clearly there can be distinct compositions of g and f. More subtly, there can be multiple 2-simplices that witness h as a composition of f and g. If f and g are a pair of composable morphisms in an ordinary category $\mathcal{C}$, we would hope that there is both a unique composition and a unique witness of that composition when viewing f and g are morphisms of $N(\mathcal{C})$. The following proposition shows this, and more generally shows that composition is unqiue up to homotopy.

Proposition 3.2.0.2.

1. *If $\mathcal{C}$ is an ordinary category, then for any morphisms $f\colon x \to y$ and $g\colon y \to z$ of $N(\mathcal{C})$, there is a unique composition of g and f in $N(\mathcal{C})$, and there is a unique 2-simplex σ witnessing the composition.*

2. *If $\mathcal{D}$ is an ∞-category and $f\colon x \to y$ and $g\colon y \to z$ are morphisms of $\mathcal{D}$, then there exists at least one composition of g and f.*

3. *With $\mathcal{D}$, f, and g as above, given two compositions of g and f there is a homotopy from one to the other.*

4. *With $\mathcal{D}$, f, and g as above, if h is a composition of g and f and there exists a homotopy from h to h', there exists a 2-simplex witnessing h' as a composition of g and f.*

Proof.

1. Trivial by the uniqueness in the extension property of the nerve.

2. We can form the horn $\sigma_0 \colon \Lambda^2_1 \to \mathcal{D}$ corresponding to the tuple $(g, *, f)$. There is an extension to $\sigma \colon \Delta^2 \to \mathcal{D}$, and the 2-simplex corresponding to σ witnesses that $\sigma(\delta_1(\Delta^2))$ is a composition of f and g.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "\sigma(d_1(\Delta^2)))"'] & & z \end{tikzcd}$$

Figure 3.6: A depiction of the horn $\sigma_0 \colon \Lambda^2_1 \to \mathcal{D}$ (left) and a depiction of extension to $\sigma \colon \Delta^2 \to \mathcal{D}$ (right).

3. Let τ_2 and τ_3 be witnesses of compositions of f and g. Then we can form the horn $\sigma_0 \colon \Lambda^3_1 \to \mathcal{D}$ specified by the tuple $(s_1(g), *, \tau_2, \tau_3)$

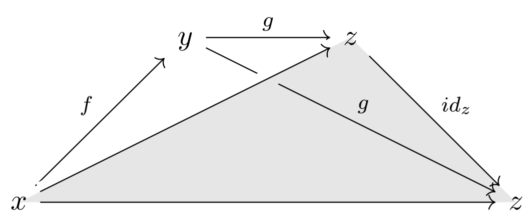

Figure 3.7: The shaded portion represents the 2-simplex that must be specified to extend this horn to a 3-simplex.

4. Suppose that σ witnesses h as a composition of g and f, and τ is a homotopy from h to h'. We form the horn $\gamma_0 \colon \Lambda^3_2 \to \mathcal{D}$ corresponding to the tuple $(\sigma, \tau, *, s_1(g))$, depicted below. We know that there is an extension $\gamma \colon \Delta^3 \to \mathcal{D}$ and $\gamma(\delta_2(\Delta^3))$ witnesses h' as a composition of f and g.

$\square$

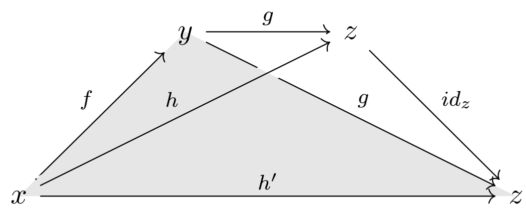

Figure 3.8: The shaded portion represents the 2-simplex that must be specified to extend this horn to a 3-simplex

Example 3.2.0.3. Given a space X, we can form the concatenation of two paths $f, g\colon [0,1] \to X$ such that $f(1) = g(0)$ in the following way, $(f * g)(t) = \begin{cases} f(2t) \text{ if } t \leq 1/2 \\ g(2t-1) \text{ if } t > 1/2 \end{cases}$. We can form a 2-simplex of $\mathrm{Sing}(X)$ that witnesses $f * g$ as a composition of g and f in the following way. Let $\sigma\colon |\Delta^2| \to X$ be defined as $\sigma(t_0, t_1, t_2) = \begin{cases} f(t_1+2t_2) \text{ if } t_0 \geq t_2 \\ g(t_2-t_0) \text{ if } t_0 \leq t_2 \end{cases}$. Any reparametrizations of this path should also be considered acceptable concatenations, and in fact one can easily form 2-simplices to witness that any reparametrization is also a composition of f and g in $\mathrm{Sing}(X)$.

Proposition 3.2.0.4. *Let $\mathcal{D}$ be an ∞-category, and $f, f'\colon x \to y$ and $g, g'\colon y \to z$. Suppose that there are homotopies σ from f to f', and τ from g to g'. Then there are homotopies between any composition of g and f and any composition of g' and f'.*

Proof. First we will show that there is a homotopy between any composition of g and f, witnessed σ_0, and any composition of g and f', witnessed σ_1. By Proposition 3.2.0.2 we know that the existence of a homotopy σ from f to f' is equivalent to the existence of a 2-simplex σ' such that $d_0(\sigma') = f$, $d_1(\sigma') = f'$, and $d_2(\sigma) = id_x$. Consider the horn $\gamma_0\colon \Lambda^3_2 \to \mathcal{D}$ corresponding to the tuple $(\sigma_0, \sigma_1, *, \sigma')$, depicted below. We know that

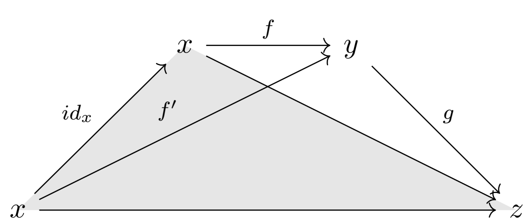

Figure 3.9: The shaded portion represents the 2-simplex that must be specified to extend this horn to a 3-simplex

this must extend to some $\gamma\colon \Delta^3 \to \mathcal{D}$, and $\gamma(\delta_2(\Delta^3))$ is a homotopy between the given composition of g and f and the given composition of g and f' (by Proposition 3.2.0.2).

One can use similar techniques to find a homotopy between a given composition (and witness of composition) of g and f' and a given composition (and witness of composition) of g' and f', which shows that there is a homotopy from any composition of g and f to any composition of g' and f' by the transitivity of homotopy. □

In all of these proofs, the fact that two morphisms are homotopic is useless with out the actual witness of composition. This highlights the fact that homotopy is a structure rather than an abstract property in ∞-categories.

Composition of morphisms in ordinary cateogories is associative, but the definition of an associative operation relies on the notion of equality. Rather, in the context of ∞-categories, we have the notion of associative upto coherent homotopy.

Proposition 3.2.0.5. *Let $\mathcal{D}$ be an ∞-category and let $f\colon w \to x$, $g\colon x \to y$, and $h\colon y \to z$ be morphisms of $\mathcal{D}$. Suppose that σ_3 witnesses p as a composition of of g and f, σ_0 witnesses q as a composition of h and g, and σ_1 witnesses r as a composition of h and p. Then there exists a 2-simplex witnessing r as a composition of q and f.*

Note that r is a composition of h with a composition of g and f, while a composition of q and f is a composition of a composition of h and g with a composition of f. Thus this proposition shows that composition is "associative upto coherent homotopy".

Proof. Consider the horn $\gamma_0\colon \Lambda^3_2$ corresponding to the tuple $(\sigma_0, \sigma_1, *, \sigma_3)$ depicted below.

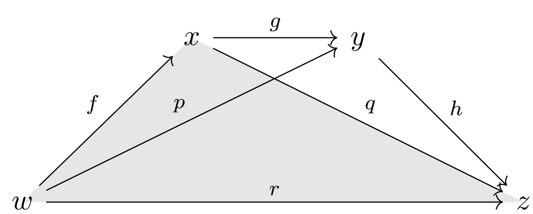

Figure 3.10: The shaded portion represents the 2-simplex that must be specified to extend this horn to a 3-simplex.

We know that there must be an extension of γ_0 to $\gamma\colon \Delta^3 \to \mathcal{D}$, and $\gamma(\delta_2(\Delta^3))$ witnesses r being a composition of q and f. □

3.3 Higher Dimensional Morphisms

In the Chapter 0 we introduced the notion of morphisms of arbitrarily high dimension. Let $\mathcal{D}$ be an ∞-category. Although it is not possible to think of every simplex of degree ≥ 2 as a morphism of that degree, it is possible in certain situations. For instance a 2-simplex that is a homotopy can be thought of as a 2-morphism from its 2nd face to its 1st face. One interpretation of Proposition 3.1.0.2 is that all of these 2-morphisms are invertible. More generally, we can think of certain n-simplices that only have two non-degenerate faces as an n-morphism from one non-degenerate face to the other (and we can think of these faces as $(n-1)$-morphisms). The existence of such an n-simplex is equivalent to the existence of another such n-simplex in which the ordering of the non-degenerate faces is reversed. In this way, all higher morphisms are invertible.

For instance, the 3-simplex depicted below can be interpreted as a 3-morphism between the two non-degenerate faces, which themselves can be thought of as 2-morphisms from f to g.

$$\begin{tikzcd}
 & y \arrow[r, "id_y"] \arrow[drr, "id_y"] & y \arrow[dr, "id_y"] & \\
x \arrow[ur, "f"] \arrow[urr, "g"] \arrow[rrr, "g"] & & & y
\end{tikzcd}$$

Figure 3.11: The interior of this simplex can be thought of as a homotopy from the 2nd to the 1st face, both of which are homotopies from f to g

3.4 The Homotopy Category

Recall from Example 2.3.0.2 the adjoint pair of functors $L\colon \mathsf{sSet} \to \mathsf{Cat}$ and $N\colon \mathsf{Cat} \to \mathsf{sSet}$. We can think of the action of L on an ∞-category $\mathcal{D}$ as quotienting by homotopy to obtain an ordinary category, an idea stated more formally in the following proposition.

Proposition 3.4.0.1. *Let $\mathcal{D}$ be an ∞-category, and let $f, g\colon x \to y$ be morphisms. Let $\overline{f}, \overline{g}\colon \overline{x} \to \overline{y}$ be the corresponding morphisms in $L(D)$. Then $\overline{f} = \overline{g}$ if and only if there exists a homotopy from f to g in $\mathcal{D}$.*

Proof. First, note that $\overline{f} = \overline{g}$ if and only if the following diagram commutes.

$$\begin{tikzcd} & \overline{y} \arrow[dr, "\overline{id_y}=id_{\overline{y}}"] & \\ \overline{x} \arrow[ur, "\overline{f}"] \arrow[rr, "\overline{g}"] & & \overline{y} \end{tikzcd}$$

Now note that by the construction given in Example 2.3.0.2 this diagram commutes if and only if there exists a 2-simplex σ such that $d_0(\sigma) = id_y$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$, which is to say that σ is a homotopy from f to g. $\square$

Let $\mathcal{D}$ be an ∞-category, and let x and y be two objects. By Proposition 3.1.0.2 we can form the set of homotopy classes of morphisms from x to y, which we denote by $\mathrm{Hom}_{h\mathcal{D}}(x, y)$. For any morphism $f : x \to y$, we denote its equivalence class by $[f]$. By Proposition 3.2.0.4, there is a well defined law

$$\mathrm{Hom}_{h\mathcal{D}}(x, y) \times \mathrm{Hom}_{h\mathcal{D}}(y, z) \to \mathrm{Hom}_{h\mathcal{D}}(x, z)$$
$$([f], [g]) \mapsto [h]$$

where x, y, and z are arbitrary objects of $\mathcal{D}$, and h is any composition of g and f. Proposition 3.2.0.5 shows that this composition law is associative. By Propositions 3.2.0.4 and 3.2.0.2 for any object d, $[id_d]$ is a left and right inverse to any composable morphism.

Thus we can define the homotopy category of $\mathcal{D}$, denoted $h\mathcal{D}$ as the category with

- objects corresponding to the objects of $\mathcal{D}$, and
- morphisms corresponding to equivalence classes of morphisms in $\mathcal{D}$, i.e. the set of morphisms from x to y in $h\mathcal{D}$ is $\mathrm{Hom}_{h\mathcal{D}}(x, y)$.

By proposition 3.4.0.1, we have that $L(\mathcal{D})$ is equivalent to $h\mathcal{D}$. This gives a more intuitive idea of the action of L on ∞-categories.

Example 3.4.0.2. Let $\mathcal{C}$ be any ordinary category. Then the homotopy category of the nerve of $\mathcal{C}$ is isomorphic to $\mathcal{C}$.

The counit of an adjunction is a natural isomorphism if and only if the right adjoint is fully faithful [Lei09]. Since the nerve is fully faithful, the counit $\varepsilon : hN \to id_{\mathsf{Cat}}$ is a natural isomorphism, and in particular, $\varepsilon_{\mathcal{C}} : hN(\mathcal{C}) \to \mathcal{C}$ is an isomorphism of categories.

3.5 Homotopy Commutative Diagrams vs Homotopy Coherent Diagrams

Definition 3.5.0.1. Let $\mathcal{D}$ be an ∞-category, and let $\mathcal{J}$ be an ordinary category. A homotopy commutative diagram in $\mathcal{D}$ indexed by $\mathcal{J}$ is a functor $F\colon \mathcal{J} \to h\mathcal{D}$.

A homotopy coherent diagram indexed by $\mathcal{J}$ is a functor of ∞-categories $N(\mathcal{J}) \to \mathcal{D}$.

Given a homotopy coherent diagram $F\colon N(\mathcal{J}) \to \mathcal{D}$, by applying L as defined in Example 2.3.0.2, we get a functor $L(F)\colon hN(\mathcal{J}) \to h\mathcal{D}$. By Example 3.4.0.2, we get a functor $\mathcal{J} \to h\mathcal{D}$. Thus, given a homotopy coherent diagram, we can construct a homotopy commutative diagram by passing to homotopy categories.

On the other hand, it is not always possible to lift a homotopy commutative diagram $\tilde{G}\colon \mathcal{J} \to h\mathcal{D}$ to some $G\colon N(\mathcal{J}) \to \mathcal{D}$ such that $L(G) = \tilde{G}$. This fact highlights the importance of making reference to specific homotopies rather than just asserting that two morphisms are homotopic. Homotopy coherent diagrams require that one specifies all of the relevant homotopies, while a homotopy commutative diagram is just one that "commutes upto homotopy" with out specifying the corresponding homotopies. However, there is non-trivial information that is lost when turning a homotopy coherent diagram into a homotopy commutative one (as evidenced that it is not always possible to lifT a homotopy commutative diagram to a homotopy coherent diagram).

The information that is lost in this process is the information encoded by the higher degree simplices, which essentially encode homotopies between homotopies. The interiors of these higher degree simplices encodes information about the coherence of the homotopies that make up its faces. This is why ∞-categories are sometimes thought of as categories up to coherent homotopy.

Example 3.5.0.2. Let $\mathcal{C}$ and $\mathcal{J}$ be any ordinary categories, and let $F\colon \mathcal{J} \to hN(\mathcal{C}) \cong \mathcal{C}$ be a homotopy commutative diagram in $N(\mathcal{C})$ indexed by $\mathcal{J}$. Since the nerve is fully faithful (Proposition 2.1.0.3), there is a corresponding lift to a homotopy commutative diagram $N(F)\colon N(\mathcal{J}) \to N(hN(\mathcal{C}) \cong N(\mathcal{C})$. Thus every homotopy commutative diagram in $N(\mathcal{C})$ can be lifted to a homotopy coherent diagram in $N(\mathcal{C})$. As such, we can say that there is no non-trivial information encoded by the higher simplices of $N(\mathcal{C})$.

3.6 Isomorphisms

There are two different approaches that one could take to defining inverses and isomorphisms in ∞-categories: one can either consider the notion of a homotopy inverse and generalize the concept to ∞-categories with the help of the singular functor, or one could simply consider if the homotopy class of a morphism is an isomorphism in the homotopy category. In line with the idea that ∞-categories are categories up to coherent homotopy, these two constructions will be equivalent.

First we consider the topological approach. Given a continuous map $p\colon A \to B$, we say that a map $q\colon B \to A$ is a left homotopy inverse of p if $q \circ p$ is homotopic to the identity map of A, id_A. Similarly, q is a right homotopy inverse of p if $p \circ q$ is homotopic to the identity map of Y. This motivates the following definition.

Definition 3.6.0.1. Let $\mathcal{D}$ be an ∞-category, and let $f\colon x \to y$, and $g\colon y \to x$.

- g is a left inverse of f if, given any composition h of g and f, there exists a homotopy from h to id_x.
- g is a right inverse of f if, given any composition h' of f and g, there exists a homotopy from h' to id_y.
- If g is both a left and a right inverse of f then we say that g is an inverse of f.
- If f has both a left and a right inverse then we say that f is an isomorphism.

By Proposition 3.2.0.2 we can see that g if a left inverse of f if and only if there is a 2-simplex witnessing id_x as a composition of g and f, and similarly g is a right homotopy inverse of f if and only if there is a 2-simplex witnessing id_y as a composition of f and g. Additionally, g is a left (or right) inverse of f exactly when f is a right (or left) inverse of g. There are a number of properties that one would desire for left and right inverses. First, if g is a left (or right) inverse of f and there exists a homotopy from g to g', it is not clear that g' is also a left (or right) inverse of f. Secondly, we want for f to be an isomorphism if and only if it is invertible. As it stands, it is possible that f has a left inverse h and a right inverse h' (making f an isomorphism), while there is no homotopy from h to h', meaning that h or h' may not be an inverse of f. Luckily for us these issues work themselves out (which we actually prove below instead of relying on the luck).

Proposition 3.6.0.2. *Let $\mathcal{D}$ be an ∞-category and suppose that $g\colon y \to x$ is a left (or right) inverse of $f\colon x \to y$, and suppose that there exists a homotopy from g to g'. Then*

g' is a left (or right) inverse of f.

Proof. Let σ witness g being a left inverse of f, and let τ be a homotopy from g to g'. Consider the horn $\gamma_0 \colon \Lambda^3_2 \to \mathcal{D}$ corresponding to the tuple $(\tau, s_1(id_x), *, \sigma)$. We know that there is an extension $\gamma \colon \Delta^3 \to \mathcal{D}$, and $\gamma(\delta_2(\Delta^3))$ witnesses that g' is a left inverse of f. The proof for right inverses is nearly identical. □

Proposition 3.6.0.3. *Suppose that $f \colon x \to y$ admits g as a left inverse and g' as a right inverse. Then there exists a homotopy from g to g'.*

Proof. Let σ_3 witness g' being a right inverse of f and let σ_0 witness g being a left inverse of f. Consider the horn $\gamma_0 \colon \Lambda^3_2 \to \mathcal{D}$ corresponding to the tuple $(\sigma_0, s_0(g), *, \sigma_3)$. Then we know that there is an extension $\gamma \colon \Delta^3 \to \mathcal{D}$ and $\gamma(d_2(\Delta^3))$ is a homotopy from g' to g. □

Propositions 3.6.0.2 and 3.6.0.3 show that if f admits a left (or right) inverse then the set of all left (or right) inverses forms a a homotopy equivalence class of morphisms. Furthermore, if f admits a left and a right inverse, then the class of the left inverses is equal to that of the right inverses. This means that f is an isomorphism if and only if it admits an inverse. In fact these statements prove a much stronger statement.

Proposition 3.6.0.4. *Let $\mathcal{D}$ be an ∞-category and let f be a morphism of $\mathcal{D}$. Then f is an isomorphism if and only if $[f]$ is an isomorphism of the homotopy category $h\mathcal{D}$. Equivalently, f admits a two-sided inverse if and only if $[f]$ does.*

Example 3.6.0.5. Let $\mathcal{C}$ be an ordinary category and let f be a morphism. Then f is an isomorphism when considered as a morphism of $N(\mathcal{C})$ if and only if f is an isomorphism when considered as a morphism of $\mathcal{C}$.

Note that equivalence classes of morphisms in $N(\mathcal{C})$ contain only one morphism. Thus if f, considered as a morphism of $N(\mathcal{C})$, has a left and a right inverse they must be equal. Thus f has an inverse when considered as a morphism of $\mathcal{C}$.

Example 3.6.0.6. Suppose that $\mathcal{D}$ is a Kan complex. Then for any morphism $f \colon x \to y$ is an isomorphism. consider the horns $\gamma_0 \colon \Lambda^2_0 \to \mathcal{D}$ corresponding to $(*, id_x, f)$, and $\gamma_0' \colon \Lambda^2_2 \to \mathcal{D}$ corresponding to $(f, id_y, *)$. There exist extensions to Δ^2 denoted γ and γ', respectively. $\gamma(\delta_0(\Delta^2))$ is a left inverse of f, while $\gamma'(\delta_2(\Delta^2))$ is a right inverse, meaning that f is an isomorphism.

In particular, this means that every morphism of the singular set of a space is an isomorphism. This aligns with our intuition as one can easily reverse paths in a space.

3.7 Functors of ∞-Categories

Our goal in this section is to motivate and define the ∞-category of functors for two given ∞-categories. We would want this construction of the functor category to "commute" with the nerve, an idea made rigorous in Proposition 3.7.0.3.

Definition 3.7.0.1. Given two ∞-categories $\mathcal{D}$ and $\mathcal{E}$, we call a simplicial map $\mathcal{D} \to \mathcal{E}$ a functor from $\mathcal{D}$ to $\mathcal{E}$. We call $[\mathcal{D}, \mathcal{E}]$ the ∞-category of functors from $\mathcal{D}$ to $\mathcal{E}$.

Proposition 3.7.0.2. *If $\mathcal{D}$ and $\mathcal{E}$ are ∞-categories then so is $[\mathcal{D}, \mathcal{E}]$.*

The proof is omitted for brevity but can be found in [Lur20, Tag 0079].

Proposition 3.7.0.3. *Let $\mathcal{B}$ and $\mathcal{C}$ be ordinary categories. Then $N(\mathcal{C}^{\mathcal{B}}) \cong [N(\mathcal{B}), N(\mathcal{C}]$.*

This proposition shows that ∞-category of functors is the correct generalization of the ordinary category of functors, as the nerve commutes with the formation of functor categories.

Proof. By the adjointness of the internal Hom functor of Cat (3.2), Proposition 2.1.0.3 (3.3), Example 2.3.0.2 (3.4), Remark 1.4.0.4 (3.5), and the adjointness of the internal Hom functor of sSet (3.5), we have the following natural isomorphism for all $n \geq 0$.

$$\begin{align} N(\mathcal{C}^{\mathcal{B}})_n &= \mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C}^{\mathcal{B}}) \tag{3.1} \\ &\cong Hom_{\mathsf{Cat}}([n] \times \mathcal{B}, \mathcal{C}) \tag{3.2} \\ &\cong \mathrm{Hom}_{\mathsf{sSet}}(N([n] \times \mathcal{B}), N(\mathcal{C})) \tag{3.3} \\ &\cong \mathrm{Hom}_{\mathsf{sSet}}(N([n]) \times N(\mathcal{B}), N(\mathcal{C})) \tag{3.4} \\ &\cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, [N(\mathcal{B}), N(\mathcal{C})]) \tag{3.5} \\ &\cong [N(\mathcal{B}), N(\mathcal{C})]_n \tag{3.6} \end{align}$$

Since we can specify a simplicial map by its actions on the set of n-simplices for each $n \geq 0$, the collection of these natural isomorphisms for all $n \geq 0$ identifies an isomorphism $N(\mathcal{C}^{\mathcal{B}}) \to [N(\mathcal{B}), N(\mathcal{C})]$ in the category sSet. $\square$

Remark 3.7.0.4. The 0-simplices of $[\mathcal{D}, \mathcal{E}]$ are functors, while the 1-simplices are called natural transformations of functors. One can think of the 2-simplices are homotopies between natural transformations and higher simplices as homotopies of homotopies.

Chapter 4

Fibrations

Fibrations of various types are common tools in both the study of homotopy theory and category theory. In this chapter, we introduce a variety of different types of fibrations of simplicial sets. The first section concerns trivial Kan fibrations, which can be used to show that the composition of morphisms in an ∞-category is unique up to a contractible space of choices. The second section works with Kan fibrations, and shows how they are the correct generalization of Serre fibrations to the context of ∞-category theory.

Definition 4.0.0.1. A simplicial morphism $p\colon X \to Y$ is a

- *Trivial Kan Fibration* if it has the right lifting property with respect to every inclusion $\delta\Delta^n \hookrightarrow \Delta^n$. That is, for every commutative diagram of solid arrows, there is a dashed arrow making the following diagram commute.

$$\begin{tikzcd}
\delta\Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

- *Kan Fibration* if it has the right lifting property with respect to every horn inclusion $\Lambda_i^n \hookrightarrow \Delta^n$. That is, for any commutative diagram of solid arrows, there is a dashed arrow making the diagram commute.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

- *Inner Kan Fibration* if it has the right lifting property with respect to every inner horn inclusion $\Lambda_i^n \hookrightarrow \Delta^n$ where $0 < i < n$.

Note that a simplicial set $\mathcal{D}$ is an ∞-category if and only if the simplicial map $\mathcal{D} \to \Delta^0$ is an inner Kan fibration.

4.1 Trivial Kan Fibrations

A CW-complex X is said to be (weakly) contractible if every map of CW-complexes $|\delta\Delta^n| \to X$ has an extension to $|\Delta^n| \to X$. Similarly, we say that a simplicial set S is contractible if every simplicial map $\delta\Delta^n \to S$ extends to a map $\Delta^n \to S$. Clearly, if a simplicial set is contractible, then its realization is weakly contractible. As is common in classical homotopy theory, we introduce a relative version of this concept.

We have already shown that composition of morphisms is unique up to homotopy in ∞-categories. We will use trivial Kan fibrations to prove a much stronger statement: the composition of two morphisms is unique up to a contractible space of choices. Proving this statement will require a substantial amount of theory development. Here we only go over the broad strokes of the proofs but a more detailed account can be found in [Lur20] and [Lur06].

Proposition 4.1.0.1. *Let $p\colon X \to Y$ be a trivial Kan fibration. Then for any object y of Y, the fiber $X \times_Y \{y\}$ is a contractible simplicial set.*

Definition 4.1.0.2. Let $\mathcal{C}$ be a category that admits pushouts, and let T be a collection of morphisms of $\mathcal{C}$.

1. T is closed under pushouts if for any pushout diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\ B' \arrow[r] & B \end{tikzcd}$$

   $f \in T$ implies that $f' \in T$.

2. Let $f\colon C \to D$ and $f'\colon C' \to D'$ be morphisms of $\mathcal{C}$. We can view these morphisms as objects in the functor category $\mathcal{C}^{[1]}$. We say that f is a retract of f' in $\mathcal{C}$ if it is a retract of f' in $\mathcal{C}^{[1]}$. The collection T is closed under retracts if f being a retract of f' and $f' \in T$ implies that $f \in T$.

3. We say that a morphism f is a transfinite composition of morphisms in T if there exists an ordinal number α and a functor $F\colon [\alpha] \to \mathcal{C}$, given by a collection of objects $\{C_\beta\}_{\beta \le \alpha}$, and a collection of morphisms $\{f_{\gamma,\beta}\colon C_\beta \to C_\gamma\}_{\beta \le \gamma \le \alpha}$, with the following properties.

   - For every non-zero limit ordinal $\lambda \le \alpha$, the functor F exhibits C_λ as the colimit of the diagram $(\{C_\beta\}_{\beta \le \lambda}, \{f_{\gamma,\beta}\}_{\beta \le \gamma \le \lambda})$.
   - For every ordinal $\beta \le \alpha$, the morphism $f_{\beta+1,\beta}$ is in T.
   - The morphism f is equal to $f_{\alpha,0}\colon C_0 \to C_\alpha$.

   We say that T is closed under transfinite composition if any transfinite composition of morphisms in T is an element of T.

If T is closed under pushouts, retracts, and transfinite composition then we say that T is weakly saturated. Given a class S of morphisms, we call the smallest weakly saturated class of morphisms containing S "the class generated by S".

The ideas of being closed under pushouts and retracts are fairly self explanatory. Transfinite composition generalizes the notion of finite composition of morphisms to an infinite setting.

Proposition 4.1.0.3. *The following sets of simplicial maps generate the same weakly saturated class.*

1. *The set of inner horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$ where $0 < i < n$.*

2. *The set of inclusions $(\delta\Delta^m \times \Delta^2)\coprod_{\delta\Delta^m \times \Lambda_1^2}(\Delta^m \times \Lambda_1^2) \hookrightarrow \Delta^m \times \Delta^2$ where $m \geq 0$.*

Proposition 4.1.0.4. *Let u be a simplicial morphism. The class of morphisms having the left lifting property with respect to u is a weakly saturated class of morphisms.*

Let $\mathcal{D}$ be an ∞-category. Then by definition, the map $q\colon \mathcal{D} \to \Delta^0$ is an inner Kan fibration. Let S denote the collection of simplicial maps that have the left lifting property with respect to q, and let S' be the weakly saturated class generated by the set of inner horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$. Since and inner horn inclusion has the left lifting property with respect to q, S' is contained in S. By Proposition 4.1.0.3 S' contains all of the inclusions $(\delta\Delta^m \times \Delta^2)\coprod_{\delta\Delta^m \times \Lambda_1^2}(\Delta^m \times \Lambda_1^2) \hookrightarrow \Delta^m \times \Delta^2$. Thus we have proved the following lemma.

Lemma 4.1.0.5. *If $\mathcal{D}$ is an ∞-category then the map $\mathcal{D} \to *$ has the right lifting property with respect to all of the inclusions $(\delta\Delta^m \times \Delta^2)\coprod_{\delta\Delta^m \times \Lambda_1^2}(\Delta^m \times \Lambda_1^2) \hookrightarrow \Delta^m \times \Delta^n$.*

Theorem 4.1.0.6. *If $\mathcal{D}$ is an ∞-category, then the restriction map $r\colon [\Delta^2, \mathcal{D}] \to [\Lambda_1^2, \mathcal{D}]$ induced by the inclusion $\Lambda_1^2 \hookrightarrow \Delta^2$ is a trivial Kan fibration.*

Given a 0-simplex σ_0 of $[\Lambda_1^2, \mathcal{D}]$ (which is just a simplicial map $\Lambda_1^2 \to \mathcal{D}$), let the corresponding tuple be $(g, *, f)$. Let $\sigma\colon \Delta^2 \to \mathcal{D}$ be a 0-simplex of $[\Delta^2, \mathcal{D}]$ such that $r(\sigma) = \sigma_0$. Then σ is a witness that $d_1(\sigma)$ is a composition of g and f. Conversely, given any witness ω of a composition of g and f, $r(\omega) = \sigma_0$. Thus we can think of the fiber $r^{-1}(\sigma_0) := \{\sigma_0\} \times_{[\Lambda_1^2, \mathcal{D}]} [\Delta^2, \mathcal{D}]$ as the the space of all choices of 2-simplices that witness a composition of g and f. Since the fibers of a trivial Kan fibration are contractible Kan complexes (Proposition 4.1.0.4), the theorem is stating that this parameter space is contractible. Thus we can safely assert that composition in ∞-categories is unique up to a contractible space of choices.

Proof. We want to show that given any commutative diagram of solid arrows, there exists a dashed arrow making the diagram commute.

Using the fact that $\mathrm{Hom}_{\mathsf{sSet}}(\delta\Delta^m, [\Delta^2, \mathcal{D}]) \cong \mathrm{Hom}_{\mathsf{sSet}}(\delta\Delta^m \times \Delta^2, \mathcal{D})$, there is a morphism $f' = ev \circ (f, id_{\Delta^2})\colon \delta\Delta^m \times \Delta^2 \to \mathcal{D}$ corresponding to f. Similarly, there is a morphism

$$\begin{tikzcd}
\delta\Delta^m \arrow[r, "f"] \arrow[d, hook, "i"] & {[\Delta^2, \mathcal{D}]} \arrow[d, "r"] \\
\Delta^m \arrow[r, "g"] \arrow[ur, dashed, "p"] & {[\Lambda^2_1, \mathcal{D}]}
\end{tikzcd}$$

$g' = ev \circ (g, id_{\Lambda^2_1})\colon \Delta^m \times \Lambda^2_1 \to \mathcal{D}$ corresponding to g. One can check that the commutativity of the solid diagram is equivalent to the condition that the restrictions of f' and g' to $\delta\Delta^m \times \Lambda^2_1$ are equal, meaning the data of f' and g' are equivalent to the data of some map $f''\colon (\delta\Delta^m \times \Delta^2) \coprod_{\delta\Delta^m \times \Lambda^2_1} (\Delta^m \times \Lambda^2_1) \to \mathcal{D}$. Thus we have shown that the data of the solid commutative diagram is equivalent to the data of the map f''.

Now suppose that there exists some simplicial map $p\colon \Delta^m \to [\Delta^2, \mathcal{D}]$. By the adjointness of the product and internal Hom, we get a corresponding morphism $p' = ev \circ (p, id_{\Delta^2})\colon \Delta^m \times \Delta^2 \to \mathcal{D}$. The conditions that $f = p \circ i$ and $g = r \circ p$ are equivalent to the condition that $f'' = p' \circ i'$ where $i'\colon (\delta\Delta^m \times \Delta^2) \coprod_{\delta\Delta^m \times \Lambda^2_1} (\Delta^m \times \Lambda^2_1) \hookrightarrow \Delta^m \times \Delta^2$. Thus we have shown that the existence of a solution to the original lifting problem is equivalent to a solution to the lifting problem depicted below.

$$\begin{tikzcd}
(\delta\Delta^m \times \Delta^2) \coprod_{\delta\Delta^m \times \Lambda^2_1} (\Delta^m \times \Lambda^2_1) \arrow[r, "f''"] \arrow[d, hook, "i'"] & \mathcal{D} \arrow[d] \\
\Delta^m \times \Delta^2 \arrow[r] \arrow[ur, dashed, "p'"] & *
\end{tikzcd}$$

By Lemma 4.1.0.5 there exists a solution to this lifting problem. □

The converse of Theorem 4.1.0.6 holds as well, meaning that ∞-categories are exactly those simplicial sets in which one can uniquely define composition of 1-morphisms up to a contractible space of choices. One can find a proof in [Lur20, Tag 0079]

4.2 Kan Fibrations

Fiber bundles make precise the idea of one space being parametrized by another. However, if one is only interested in studying spaces up to homotopy equivalence, then the requirement that all of the fibers be homeomorphic is excessive. Fibrations generalize fiber bundles by only requiring that the fibers are homotopic to each other. More concretely, a fibration is a continuous map that satisfies the homotopy lifting property with respect to any space.

If one is only interested in studying spaces up to weak homotopy equivalence, then even this definition is too strict. Serre fibrations are continuous maps that have the homotopy lifting property with respect to any CW-complex. Since for any space X, one can construct a CW-complex Y and a map $X \to Y$ that is a weak homotopy equivalence, Serre fibrations can be used to study spaces up to weak homotopy equivalence. In this section we will give a simple proof that the study of Kan fibrations in ∞-categories encapsulates the study of Serre fibrations.

Definition 4.2.0.1. A map $f\colon E \to B$ is a Serre fibration if it has the homotopy lifting property with respect to every CW-complex. That is, for any CW-complex X, and any commutative diagram indicated by the solid arrows, there exists a morphism indicated by the dashed arrow which makes the diagram commute.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook, "X\times\{0\}"'] & E \arrow[d, "f"] \\
X \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

Because CW-complexes are colimits of disks of various dimensions, one can show that only requiring the homotopy lifting property with respect to any closed disk gives an equivalent definition for Serre fibrations. Since $|\Lambda^n_i| \cong D^{n-1}$ and $|\Delta^n| \cong D^n \cong D^{n-1} \times I$, we can define Serre fibrations as maps which have the homotopy lifting property with respect to all inclusions $|\Lambda^n_i| \hookrightarrow |\Delta^n|$. That is, for any commutative diagram indicated by the solid morphisms, there exists a morphism indicated by the dashed morphism making the diagram commute.

$$\begin{tikzcd}
{|\Lambda^n_i|} \arrow[r] \arrow[d, hook] & E \arrow[d, "f"] \\
{|\Delta^n|} \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

By the adjointness of the singular and geometric realization functors, applying the singular functor to a Serre fibration gives a Kan fibration. In other words (or rather diagrams), if $f\colon E \to Y$ is a Serre fibration, then there exists a dashed morphism making the following diagram commute.

$$
\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & \mathrm{Sing}(E) \arrow[d, "\mathrm{Sing}(f)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Sing}(B)
\end{tikzcd}
$$

Thus the study of Kan fibrations of ∞-categories encapsulates the study of Serre fibrations of topological spaces.

Chapter 5

Conclusion

The goal of this project was to show that our definition of an ∞-category models the notion of a category that has morphisms of degree n between morphisms of degree $n-1$, for all integers $n \geq 1$. This allows us to never consider the equality of objects or morphisms and instead only consider if there is an isomorphism of degree $n+1$ between two morphisms of degree n.

By considering the singular functor, we showed that the study of Kan complexes, which are a particular type of ∞-category, is equivalent to the study of the homotopy theory of spaces. We also showed that the nerve is fully faithful, meaning that the study of ∞-categories encapsulates the study of ordinary categories.

As a brief example of the utility of ∞-category theory, we considered two different types of fibrations. Trivial Kan fibrations were used to show that the composition of morphisms in an ∞-category is unique up to a contractible space of choices. In addition, we noted that converse is true, meaning that ∞-categories are exactly those simplicial sets where composition is uniquely defined up to a contractible space of choices. Our discussion around Serre fibrations gave an example of how the study of homotopy theory translates to the study of ∞-category theory.

Further Readings

- Although Chapter 3 generalized many categorical constructions to the setting of ∞-categories, a notable omission is the ∞-category analogue of limits and colimits, which was left out for brevity. More information can be found in 1.2.13 of [Lur06].

- In this project we explored only two examples of ∞-categories: the singular sets of spaces, and the nerves of categories. However, there are many other useful examples of ∞-categories. A discussion of some of these examples can be found in [Lur20, Tag 007J].
- There are other approaches to higher category theory. In this project we considered an approach based on simplicial sets, but another common approach is based on topological categories. These two approaches are in fact equivalent. For the definition of a topological category as well as a proof of this equivalence, one can consult 1.1.3 of [Lur06].
- An important area of research in higher category theory regards translating proofs from one model to another. For instance, [RV19] shows the equivalence of certain basic constructions between various models for higher categories. There are also various efforts to formulate a truly model-independent approach to higher category theory. An intuitive explanation of the goals of such an approach can be found in the lecture [Ras].

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