Combinatorics Final Project

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Introduction

A classic problem in combinatorics is counting the number of ways to permute $[n]$. We can think of a permutation of $[n]$ as a set of ordered pairs where the first coordinate of each pair is the element and the second coordinate is the position of that element in the permutation. For example, the permutation $(3, 1, 2)$ corresponds to the set $\{(1, 2), (2, 3), (3, 1)\}$

This leads to a visualization of the possible permutations using a $n \times n$ board, where the rows represent the elements and the columns represent the position. Specifically, every permutation corresponds to a set of n squares of the board, no two of which lie in the same row or column. The fact that no two lie in the same row indicates that an element can occupy only one position in the permutation. The fact that no two squares can be in the same column represents that there can only be one element in each position.

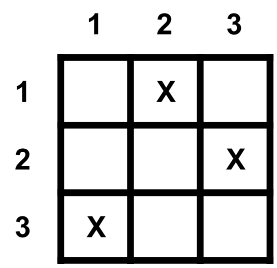

Figure 1: The representation of the permutation $(3, 1, 2)$ on a 3×3 board.

Picking such squares is equivalent to placing n non-attacking rooks on the board. For those unfamiliar with chess, a rook is a piece that can move along the row and column that it is currently in. We say that two rooks are non attacking if they are not in the same row or column (meaning they can not capture each other).

We can also build certain conditions into the board by removing some squares from the board. For example, if we do not want the element 2 to be in the third position ever, we can remove the square $(2, 3)$ from the board. This is (kind of) the motivation for studying the number of ways to place non-attacking rooks on various boards.

1 Rook Polynomials of Ferrers Boards [3]

In class we studied the Ferrers board of partitions of n in order to count the number of certain classes of partitions. We can also think of the Ferrers board as a board upon which we can place non-attacking rooks.

Definition: Ferrers Board. For a partition $\mu = (\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots \geq \mu_k)$ of n (with all $\mu_i > 0$), we define the Ferrers board of μ, F_μ to be the left justified rows of squares where the first row has μ_1 squares, the second has μ_2 squares, and the ith row has μ_i squares.

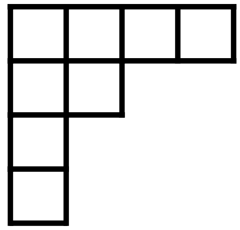

Figure 2: The Ferrers board for the partition $(4,2,1,1)$

Definition: Rook Polynomial of Partition. Given a partition μ, we define $r_k(\mu)$ to be number of ways to place k non-attacking rooks on F_μ. Let $r_0(\mu) = 1$. The rook polynomial of μ is

$$R_\mu(x) = \sum_{k=0}^{\infty} r_k(\mu) x^k$$

For example, using the previous permutation $\mu = (4,2,1,1)$, we see that $r_1(\mu) = 8$, $r_2(\mu) = 14$, and $r_3(\mu) = 4$. This means that $R_\mu(x) = 1 + 8x + 14x^2 + 4x^3$

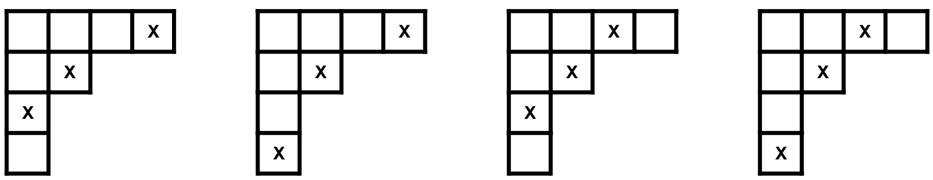

Figure 3: The four ways to place three non attacking rooks on $F_{(4,2,1,1)}$

We want to explore when two partitions of n have the same rook polynomial.

Definition: Rook Equivalence. Two partition μ and ν are rook equivalent if and only if they have the same rook polynomial, which means that $r_k(\mu) = r_k(\nu)$ for all $k \geq 0$.
Since $r_1(\mu) = |\mu|$ for any μ, for μ and ν to be rook equivalent it must be true that $|\mu| = |\nu|$. This is a necessary but not sufficient condition.

Theorem: Rook Equivalence of Partitions. Suppose that μ and ν are partitions of n. Write them in the form $\mu = (\mu_1 \geq \cdots \geq \mu_n)$, $\nu = (\nu_1 \geq \cdots \geq \nu_n)$ by adding zero parts to the end as needed. Then $R_\mu(x)$=$R_\nu(x)$ are equal if and only if

$$[\mu_1 + 1, \mu_2 + 2, \cdots, \mu_n + n] = [\nu_1 + 1, \nu_2 + 2, \cdots, \nu_n + n]$$

Proof. The idea of the proof is to use a different basis for the vector space of polynomials. First we will show that two rook polynomials are equal under the different basis if and only if they are equal in the standard basis. Then we will show that two rook polynomials are equal in the different basis if and only if the multisets mentioned in the theorem are equal. But first we need some definitions.

Definition: Vector Space (of Polynomials). A formal definition is not useful for understanding so we decided to give a less formal one. A vector space is a space that is closed under addition of vectors and multiplication of vectors by scalars. The space of polynomials is a vector space because we can add polynomials and multiply and of polynomials by a scalar.

Definition: Basis of a Vector Space. A basis for a vector space is a way to represent any element of a vector space as a linear combination of the basis elements. For the vector space of polynomials, the most common basis is the monomial basis $\{x^n : n \geq 0\}$. For example, the polynomial $3x^2 + 4x + 6$ is represented by $(6, 4, 3)$ with respect to the monomial basis.

Definition: The Falling Factorial Basis. The falling factorial $(x)\downarrow_n$ is defined to be$(x)(x-1)\cdots(x-n+1)$. The falling factorial basis is the set $\{(x)\downarrow_n : n \geq 0\}$.
Under the falling factorial basis, every polynomial has a unique representation because of the fact that the falling factorial basis is linearly independent. (To show this arrange the basis elements in a matrix and note that it is row reducible to the identity.) Specifically, if we start with a polynomial represented in the monomial basis, then there is exactly one corresponding representation in the falling factorial basis.
This means that two polynomials' representations are equal in the monomial basis if and only if they are equal under the falling factorial basis.
The representation of a rook polynomial of a partition λ in the falling factorial basis is

$$R'_\lambda(x) = \sum_{k=0}^{n} r_{n-k}(x)\downarrow_k = \sum_{k=0}^{n} r_{n-k}(x)(x-1)\cdots(x-k+1)$$

We will show that for two partitions, μ and ν, $R'_\mu(x) = R'_\nu(x)$ if and only if the multisets in the theorem are equal. To do this we will use rook polynomials. Consider the extended board $F_\mu(x)$ where we add x to every part.

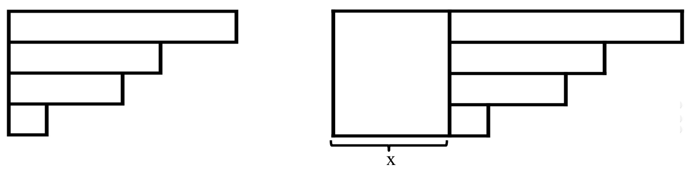

Figure 4: On the left is the original board F_μ and on the right is the extended board $F_\mu(x)$

We will find the number of ways to place n non attacking rooks on this extended board in two ways. The first will show that it is the same as our formula for $R'_\mu(x)$. The second way will give us a convenient way to show that the formulas for two different partitions are equal if and only if the multisets in the theorem are equal.

First, we assume that exactly k of the rooks are on the original board F_μ, then we know that the remaining $n-k$ of the rooks must be placed in the unused $n-k$ rows, and since none of these additional rooks can be on the original F_μ they must all be in the first x columns of the board. This means that there are $x(x-1)\cdots(x-(n-k)+1) = (x)\downarrow_n$ ways to place the rest of the rooks. Summing over all k, we get that the number of placements of n rooks is $\sum_{k=0}^{n} r_k(\mu)(x)\downarrow_{n-k}$. If we replace k with $n-k$ then the total sum will stay the same and we get that the number of ways to place n non attacking rooks on $F_\mu(x)$ is :

$$\sum_{k=0}^{n} r_{n-k}(x)\downarrow_k = R'_\mu(x)$$

On the other hand, we can count the number of ways to place n non attacking rooks on $F_\mu(x)$ by first placing a rook on the last row, and then placing a rook on the second to last row and so on. Each time we place a rook, we eliminate one column that can be used in the placement of the following rooks. There are $x+(\mu_n)$ ways to place the first rook, $x+(\mu_{n-1}-1)$ ways to place the second, and so on. For an arbitrary row μ_i there are $x+(\mu_i-(n-i))$ places to put a rook. We get that the total number of ways to place n non attacking rooks is

$$\prod_{i=1}^{n} x+(\mu_i-(n-i))$$

Now we know that

$$R'_\mu(x) = \prod_{i=1}^{n} x+(\mu_i-(n-i))$$

$$R'_\nu(x) = \prod_{i=1}^{n} x + (\nu_i - (n-i))$$

Before we can deliver the punch line, we need to go over some more stuff about polynomials. We call a polynomial prime if it can not be written as the product of other polynomials (I think that this is technically the definition for irreducible, but in this ring I think that the two are the same). It is known that there is only one way to decompose a polynomial in one variable with real coefficients into the product of prime polynomials (much like how there is only one way to write an integer as the product of prime integers).

Now we note that each term in the products above is of the form $x + c$ for some real constant c. These terms are clearly prime. So the product is a decomposition of $R'_\mu(x)$ into the product of primes. This means that $R'_\mu(x) = R'_\nu(x)$ if and only the products associated with each have the same terms (up to reordering). This is equivalent the statement to $[x + (\mu_i - (n-i)] = [x + (\nu_i - (n-i)]$. If we subtract x and n from every term of the multisets we get that $[\mu_i + i] = [\nu_i + i]$, which is what we wanted to prove. $\square$

Example The partitions of (2, 2, 0, 0) and (3, 1, 0, 0) are rook-equivalent, because [3, 4, 3, 4] = [4, 3, 3,4]. On the other hand, the partitions (4, 2, 1) and (5, 2) are not rook-equivalent, since $[5, 4, 4, 4, 5, 6, 7] \neq [6, 4, 3, 4, 5, 6, 7]$.

2 Matching and Rook Polynomials [2]

In this section we will use graph theory to solve some rook polynomial problems. First we will consider the matching polynomial of a graph. Then we will find some recurrences, use these to prove some relationships between a graph and its complement. Finally, we will apply all of this to rook polynomials.

Definition: r-Matching of a Graph. An *r-matching* in a graph G is a set of r edges, no two of which have a vertex in common. We denote the number of r matching by $p(G, r)$.

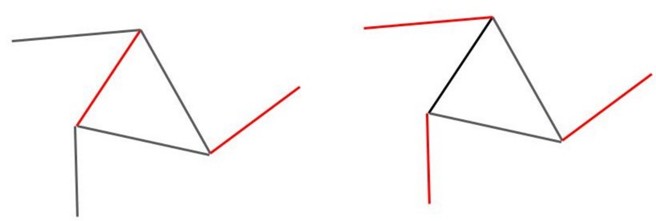

Figure 5: Examples of a 2-matching and a 3-matching

The Matching Polynomial of a Graph. For a graph G with n vertices, the matching polynomial is

$$\mu(G, x) := \sum_{r=0}^{\infty} (-1)^r p(G, r) x^{n-2r}$$

Note that in this definition, the power of x is the number of vertices not included in the r-matching that are counted.

2.1 Recurrences

Here we are interested in finding the matching polynomial for a graph by using the matching polynomials of various sub graphs. We have the following (big ol') theorem that gives us a few recurrences.

Theorem. The matching polynomial satisfies the following identities:

1. $\mu(G \cup H, x) = \mu(G, x)\mu(H, x)$
2. $\mu(G, x) = \mu(G \setminus e, x) - \mu(G \setminus uv, x)$ where $e = \{u, v\}$ is an edge in G
3. $\mu(G, x) = \mu(G \setminus u, x) - \sum_{i \;\; u} \mu(G \setminus ui, x)$

Proof. (1) Note that any r-matching in $G \cup H$ is an s-matching in G along with a $r - s$-matching in H. The formal proof is simple and can be found in [2] but is not insightful so we left it out. □

Proof. (2) follows from analyzing the two cases for an r-matching: either it contains the edge e and has an $r-1$-matching on the rest of the graph (meaning the graph without the points u and v), or it does not contain the edge e in which case it is an r-matching of the graph with out the edge e. We left out the formal proof again. □

Proof. (3) Similar to the previous proof. We condition on whether or not the r-matching contains an edge which contains the vertex u. □

2.2 Integrals

This section will explore the relationship between the matching polynomial of a graph and its compliment.

Compliment of a Graph. For a graph G its complement, $\overline{G}$, is the graph with the same vertex set but the complement of the edge set. This means that all the vertices that are not adjacent in G are adjacent in $\overline{G}$ and all those that are adjacent in G are not in $\overline{G}$.

Perfect Matching of a Graph. A perfect matching in a graph is a set of edges such that every vertex appears in exactly one edge. Less formally, one can think of a perfect matching (if it exists) as a maximal r-matching. We denote the number of perfect matching of G with $\mathrm{pm}(G)$.

We will find a relation between the number of perfect matching of a graph, and the matching polynomial of its complement. Interestingly, this relation will involve integrals. In order to prove this relation we first need to establish a recurrence for the number of perfect matching of a graph. Then we will find a formula for the number of perfect of K_n and show that this formula equals a certain integral. Finally, we will use this integral to prove the relation mentioned at the start of this paragraph.

Proposition. $\mathrm{pm}(\overline{G}) = \mathrm{pm}(\overline{G \setminus e}) - \mathrm{pm}(\overline{G} \setminus uv)$ where $e = \{u, v\}$ is an edge in G.

Proof. Consider $\mathrm{pm}(\overline{G \setminus e})$, the number of perfect matching of $\overline{G \setminus e}$. There are two disjoint cases for a perfect matching. If e is in the perfect matching, then the rest of the perfect matching is a perfect matching of $\overline{G \setminus e} \setminus uv = \overline{G} \setminus uv$, because we have already included the vertices u and v in the matching. On the other hand, if e is not in the perfect matching, then the matching is a perfect matching of $\overline{G \setminus e} \setminus e = \overline{G}$. This means that $\mathrm{pm}(\overline{G \setminus e}) = \mathrm{pm}(\overline{G} \setminus uv) + \mathrm{pm}(\overline{G})$. □

Now we will determine a formula for the number of perfect matching of K_{2m}. We start by finding a formula for the number of r-matching of K_n, and then note that any perfect matching of K_{2m} is a m-matching.

Proposition: $p(K_n, r) = \binom{n}{2r} p(K_{2r}, r)$

Proof. To prove this we will count the number of ways to make an r-matching of K_n. Any r-matching of K_n involves $2r$ vertices, and there are $\binom{n}{2r}$ ways to pick these vertices. Now we need to pick which edges we will choose such that they cover each of these 2r vertices exactly once. The subgraph made with these $2r$ vertices is K_{2r}, and we need an r-matching of this subgraph, and there are $p(K_{2r}, r)$ ways to do this. □

Now we want to find a formula for p(K_{2r}, x). We start by finding a recurrence. Since any r matching of K_{2r} contains all of the vertices of K, we can build them by starting with a vertex, followed by choosing one of the edges containing this vertex. There are $2r-1$ such edges. Now we need a $r-1$-matching of the remaining $2r-2$ vertices. Since the subgraph of the remaining vertices is K_{2r-2}, there are $p(K_{2r-2}, r-1)$ ways to do this. This gives us

$$p(K_{2r}, r) = (2r-1)p(K_{2r-2}, r-1)$$

Proposition: p$(K_{2r}, r) = \frac{(2r)!}{r!2^m}$

Proof. by induction on r:

(Basis) r=1: $p(K_2, 1) = \dfrac{(2)!}{(1!)(2^1)} = 1$

(Induction) $p(K_{2r}, r) = (2r-1)p(K_{2(r-1)}, r) = (2r-1)\dfrac{2(r-1)!}{(r-1)!2^{r-1}} = \dfrac{(2r-1)!}{(r-1)!2^{r-1}}$ By multiplying the numerator and denominator by 2r, we get $\dfrac{(2r)!}{r!2^r}$

□

We now know that number of perfect matching in K_{2m} is given by

$$\frac{(2m)!}{m!2^m} = p(K_{2m}, m) = \frac{(2m)(2m-1)(2m-2)(2m-3)...(3)(2)(1)}{m(m-1)(m-2)(m-3)...(3)(2)(1)}$$

We can simplify by factoring 2 out of every other term (of which there are m), to get $(2m-1)(2m-3)(2m-5)...(3)(1)$

Lemma: The number of perfect matching in K_n is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^{-x^2/2}} x^n dx$$

Proof. We set the integral equal to $M(n)$. Then, using integration by parts,

$$M(n) = \frac{1}{\sqrt{2\pi}} \left[\frac{x^{n+1}}{n+1} e^{-x^2/2} \right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x^{n+2}}{n+1} e^{-x^2/2} dx$$

Notice that the first part becomes 0. Then, we are left with what looks similar to $M(n)$ but here, the power of x is n+2 and there is a denominator n+1. We thus have $M(n) = \dfrac{M(n+2)}{n+1}$. Substituting n by n-2, we get $M(n-2) = \dfrac{M(n)}{n-1}$ => $M(n) = (n-1)M(n-2)$. We know that $M(1) = 0$ and $M(0) = 1$, so whenever we have an odd value for n, say 5, we get $M(5) = 4 \cdot 2 \cdot M(1)$which is 0. However, if n is even, say 6, $M(6) = 5 \cdot 3 \cdot 1 \cdot M(0)$ Notice that the resulting value is the product of all the odd numbers less than n. If n = 2m, we can rewrite the $M(n)$ as

$$M(2m) = (2m-1)(2m-3)(2m-5)...3.1 = pm(K_n)$$

□

Theorem. For any graph G:

$$pm(\overline{G}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \mu(G, x) dx$$

Proof. By induction on the number of edges. For the base case consider the graph G with n vertices that has no edges. Then $\overline{G} = K_n$. Since $\mu(G, x) = x^n$, the integral becomes $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-x^2/2} x^n$. By the previous lemma, this is pm(K_n) =pm$(\overline{G})$, so the result holds.

Now assume that G has some edges and that the result holds for any subgraph of G. Let us call the integral on the right side of the equation I(G). We know that $\mu(G,x) = \mu(G \setminus e, x) - \mu(G,x)$. This means that I($G$)=I($G \setminus e$)-I($G \setminus uv$). By the inductive hypothesis, the right side of this equation is pm($\overline{G \setminus e}$)−pm($\overline{G \setminus uv}$) =pm($\overline{G \setminus e}$)−pm($\overline{G} \setminus uv$). We know that this is equal to pm($\overline{G}$) by the proposition, so we have proved the theorem. ∎

Now we are going to consider the number of perfect matching of a bipartite graph. First, note that $\overline{K_m \cup K_n} = K_{m,n}$.

Proposition: $pm(K_{m,n}) = \{ {}^{m! \text{ if } m=n}_{0 \text{ if } m \neq n}$

Proof. Suppose for the sake of contradiction that when $m \neq n$ the number of perfect matching is not 0. Without loss of generality, suppose that $m < n$. Since all of the edges in a bipartite graph go from a vertex of type n to a vertex of type m, there must be n edges in a perfect matching (in order to cover all of the n type vertices). However, every edge contains a vertex of type m, so by the pigeonhole principle, there must be two edges that cover the same m type vertex. This is a contradiction because this is no longer a perfect matching.
Now suppose that $m = n$. Then we can build the perfect matching by first choosing which edge we want to use to cover the first of the n type vertices. There will be m choices for this. For the edge that covers the second n type vertex we will have $m - 1$ choices because the edge between the second n type vertex and the first chosen m type vertex can not be chosen. Repeating this shows that the number of perfect matching is m! ∎

Now we can further note that since

$$pm(K_{m,n}) = \begin{cases} m! \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$$

and

$$pm(K_{m,n}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \mu(K_m \cup K_n, x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \mu(K_m, x)\mu(K_n, x) dx$$

We get

$$pm(K_{m,n}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \mu(K_m, x)\mu(K_n, x) dx = \begin{cases} m! \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$$

Since the matching polynomial for every complete graph K_n is a monic polynomial of degree n, they are linearly independent and form a basis for the vector space of polynomials. Specifically we can write the matching polynomial for any graph as a linear combination of the matching polynomials for complete graphs:

$$\mu(G, x) = \sum_{r=0}^{\infty} c_r \mu(K_r, x) \text{ for some } c_r \in \mathbb{R}$$

We are interested in finding these c_r. This will give us a convenient way to express the matching polynomial of any graph in terms of polynomials that are easier to compute. To do this we first define the bilinear form

$$(p, q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} p(x) q(x) dx$$

Then, $(\mu(G,x), \mu(K_r,x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\sum_{i=0}^{\infty} c_i \mu(K_i, x) \right] \mu(K_r, x) dx$

Using the formula for bipartite graphs from above, we get that this is the same as $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_r \mu(K_r, x) \mu(K_r, x) dx = c_r r!$ According to the previous theorem, this quantity is also equal to $pm(\overline{G \cup K_r})$.
Now we want to find a formula for $pm(\overline{G \cup K_r})$ so that we can solve for c_r

$\overline{G \cup K_r}$ looks like the complement of G along with r vertices that are not connected to each other but are connected to each vertex of $\overline{G}$. So we can think of a perfect matching as a $n-r$ matching of $\overline{G}$ along with a choice of r edges that connect the vertices of $\overline{K_n}$. So in total, we get that the number of perfect matching of $\overline{G \cup K_r}$ is

$$\mathrm{p}(\overline{G}, \frac{n-r}{2})r!$$

If $\frac{n-r}{2}$ is not an integer then we we take the number of perfect matching to be 0. So we have proved the following theorem:

Theorem: $\mu(G,x) = \sum_{r=0}^{n} p(\overline{G}, \frac{n-r}{2})\mu(K_r, n) = \sum_{n=0}^{\lfloor n/2 \rfloor} p(\overline{G}, m)\mu(K_{n-2m}, x)$

The take away from this theorem is that the matching polynomial of G is determined by the matching polynomial of $\overline{G}$

2.3 Rook Polynomials

Here we will use all of the graph theory that we have developed to this point to address rook polynomials. We can think of a $n \times m$ board as the bipartite graph $K_{n,m}$ where the n type vertices correspond to the rows and the m type vertices correspond to the columns. An edge in this graph corresponds to a square on the board. A lot of the time we are not interested in the full $n \times m$ board, but some sub-board. A *spanning subgraph* is a subgraph whose edges contain all of the vertices of the original graph. A spanning subgraph of our bipartite graph corresponds to a board in which every row and every column has at least one square. This motivates the following definition for the *rook polynomial* of a spanning subgraph G of $K_{n,n}$:

$$\rho(G,x) := \sum_{r=0}^{n} (-1)^r p(G,r) x^{n-r}$$

Note here that if we have an r-matching of G then it corresponds to a placement of r non-attacking rooks because none of the chosen edges share vertices which means that none of the corresponding squares share rows or columns.

It is clear that $\rho(G,x^2) = p(G,x)$. This means that all of the results from the first theorem about the recurrences for matching polynomials translate directly to the rook polynomial.

Definition: Bipartite Complement. If G is a spanning subgraph of $K_{n,n}$ then we define the bipartite complement, $\tilde{G}$ to be the graph with the same vertices but with those edges in $K_{n,n}$ that are not in G.

Using the pieces that we have been building up one can show

Theorem: Let G be a spanning subgraph of $K_{n,n}$ then the number of perfect matching of $\tilde{G}$ is equal to

$$\int_0^\infty \rho(G,x)e^{-x}dx$$

We can use this theorem to find easy solution to two classic combinatorics problems. The first is counting the number of derangement of $[n]$. Clearly this is equivalent to placing n rook on an $n \times n$ board with the main diagonal removed. Let B be this board, and $G(B)$ be the associated bipartite graph. If we let nK_2 the the union of n graphs K_2, then we see that $G(B)$ is the complement in $K_{n,n}$ of nK_2. Now we need to find the number of perfect matching in $G(B)$ (which corresponds to arrangements of n non attacking rooks on B).

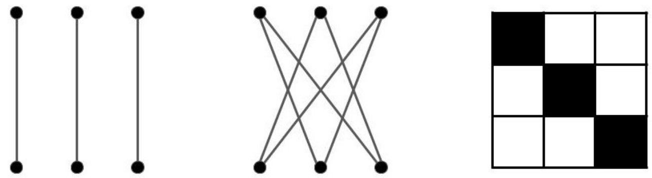

Figure 6: From left to right: $3K_2$, the complement of $3K_2$ in $K_{3,3}$, and the corresponding board (where removed squares are blacked out). Any placement of 3 non attacking rooks on the board corresponds to a derangement of $[n]$.

The rook polynomial of nK_2 is clearly $(x-1)^n$. Then the theorem implies that the number of derangement of $[n]$ is

$$\begin{aligned}\int_0^\infty \rho(nK_2,x)e^{-x}dx &= \int_0^\infty (x-1)^n e^{-x}dx \\ &= \int_1^\infty (x-1)^n e^{-x}dx + \int_0^1 (x-1)^n e^{-x}dx \\ &= \frac{n!}{e} + R_n\end{aligned}$$

We have just set the second integral to be R_n with out actually integrating this part. Now we can note that for any value between 0 and 1 (the bound of integration) $e^{-x} \geq 1$. This implies that the absolute value of the integral is less than $\int_0^1 (x-1)^n$. So we get that

$$|R_n| < \int_0^1 (x-1)^n = \frac{1}{n+1}$$

This means that for large enough n, the number of derangement of $[n]$ is the integer closest to $\frac{n!}{e}$. That is so cool that I wanted to end the sentence with an exclamation point but decided that it would cause too much mathematical confusion!

Another classic problem is the *Menage* problem. Suppose that we have n married couples. We want to seat these couples around a circular table, alternating male and female, and ensure that no one sits next to their spouse.

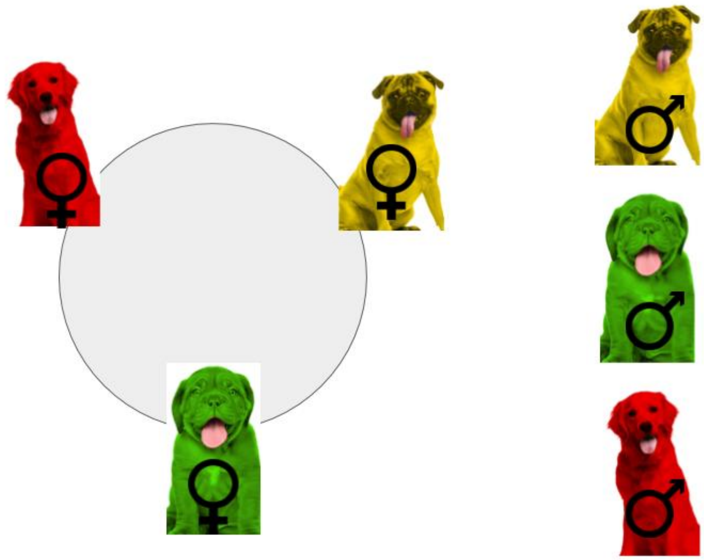

Figure 7: A (very cute) depiction of the Menage problem for $n = 3$

We can first seat the female dogs in any order and then number them clockwise from 1 to n. Then the valid arrangements correspond to permutations of the male dogs with the conditions that $\pi[i] \notin \{i, i-1\}$. The bipartite complement of the graph is the cycle C_{2n}

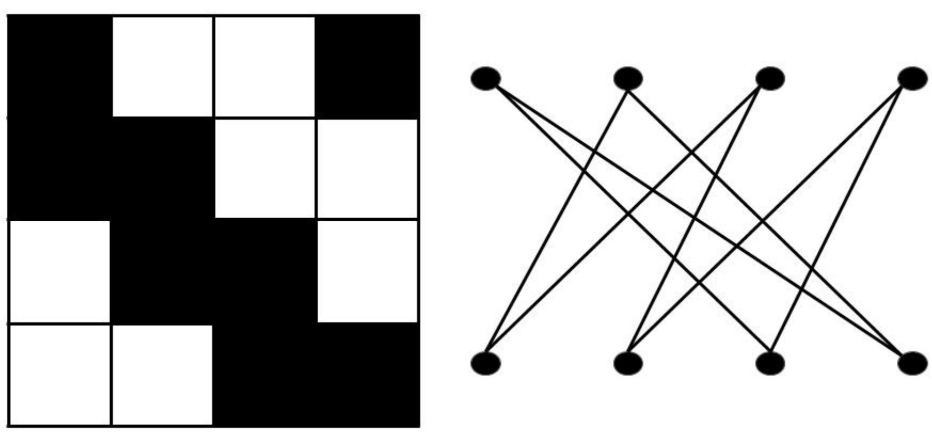

Figure 8: The board and bipartite complement of the graph associated with the Menage problem for $n = 4$

We can show that $p(C_{2n},r) = 2n\binom{2n-r}{r}/(2n-r)$. This means that the number of valid seatings is

$$\begin{aligned}\int_0^\infty \rho(C_{2n},x)e^{-x}dx &= \sum_{r=0}^{n}(-1)^r p(C_{2n},r)(n-r)! \\ &= \sum_{r=0}^{n}(-1)^r \frac{2n}{2n-r}\binom{2n-r}{r}(n-r)!\end{aligned}$$

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